

THE SIXTH PAINLEVÉ TRANSCENDENT AND UNIFORMIZATION OF ALGEBRAIC CURVES

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ABSTRACT. We exhibit a remarkable connection between sixth equation of Painlevé list and infinite families of explicitly uniformizable algebraic curves. Fuchsian equations, congruences for group transformations, differential calculus of functions and differentials on corresponding Riemann surfaces, Abelian integrals, analytic connections (generalizations of Chazy's equations), and other attributes of uniformization can be obtained for these curves. As byproducts of the theory, we establish relations between Picard–Hitchin's curves, hyperelliptic curves, punctured tori, Heun's equations, and the famous differential equation which Apéry used to prove the irrationality of Riemann's $\zeta(3)$.

Key words and phrases. Painlevé-6 equation, Picard–Hitchin solutions, algebraic curves of higher genera, Jacobi's theta-functions, automorphic functions, Fuchsian equations, Apéry's differential equation, analytic connections on Riemann surfaces.

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1. INTRODUCTION

The first example of general solution to the famous sixth Painlevé transcendent

$$\begin{aligned} \mathcal{P}_6 : \quad y_{xx} = & \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) y_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y_x \\ & + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left\{ \alpha - \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} - \left(\delta - \frac{1}{2} \right) \frac{x(x-1)}{(y-x)^2} \right\} \end{aligned} \quad (1)$$

was obtained in 1889 before the equation (1) itself had been derived by Richard Fuchs in 1905 [29]. This case corresponds to parameters $\alpha = \beta = \gamma = \delta = 0$ and is referred frequently to as Picard's solution [56]. Surprisingly, but the second one was obtained by N. Hitchin [36] after more than one hundred years. It corresponds to parameters $\alpha = \beta = \gamma = \delta = \frac{1}{8}$. Presently these solutions are the only instances, up to automorphisms in the space $(\alpha, \beta, \gamma, \delta)$, when solution of (1) is known in its full generality.

One year after Fuchs, Painlevé [54] gave a remarkable form to (1) which is known nowadays as the \wp -form of \mathcal{P}_6 -equation. The modern representation of this result is given by the nice equation obtained independently by Babich & Bordag [4] and Manin [51]:

$$-\frac{\pi^2}{4} \frac{d^2 z}{d\tau^2} = \alpha \wp'(z|\tau) + \beta \wp'(z-1|\tau) + \gamma \wp'(z-\tau|\tau) + \delta \wp'(z-1-\tau|\tau). \quad (2)$$

1.1. Motivation. It is not a matter of common knowledge that original motivation of Picard and Painlevé, when deriving their results, was closely related to construction of single-valued analytic functions. Painlevé himself repeatedly wrote (see his *Oeuvres* [55]) about representing solutions in terms of single-valued functions and Picard mentions it about throughout his almost 200-page-long treatise [56]. See also the survey article by R. Conte in [18, p. 77–180] and book [19] wherein single-valuedness of functions are constantly emphasized. It is also known that original statement of the problem on fixed critical singularities in solutions of ordinary differential equations (ODEs) was initiated by Picard himself [56, Ch. V] and subsequently developed by Painlevé and Gambier [31].

On the other hand, it has long been known that equation \mathcal{P}_6 is a rich source of algebraic solutions

$$F(x, y) = 0 \quad (3)$$

and genera of corresponding Riemann surfaces \mathcal{R} can be made as great as is wished. Effective description of such surfaces and single-valued functions on them is the subject of the theory of uniformization of algebraic curves and automorphic functions [44, 28, 8]. In this theory, the algebraic irrationality (3) is completely determined by the two principal transcendental meromorphic objects on \mathcal{R} . These are function field generators $x(\tau)$ and $y(\tau)$, of which all other meromorphic functions are built: rational functions $R(x, y)$, differentials, Abelian differentials, and their integrals. Fundamentally, the constructive ‘meromorphic analysis’ on some \mathcal{R} can be thought of as solved problem if we have at our disposal the *constructive* representation for *both* of these functions. The ‘constructive’ means here that they can be manipulated analytically (differentiation, integration, etc) like rational or elliptic functions.

First examples of algebraic solutions to equation (1) were obtained by R. Fuchs in work [30] which is less known than his 1907 work in Math. Annalen with the same title as [30]. These solutions correspond to Picard's case of parameters. Since the late 1990's list of algebraic solutions to \mathcal{P}_6 came into rapid growth thanks to works by Dubrovin, Mazzocco, Kitaev, Boalch, Vidūnas, Hitchin himself, and others. Complete reference list to this topic would be rather lengthy (see, e.g., works [52, 61, 11]) but recent works [47] and [12] contain already classification results concerning all the algebraic solutions (a relevant analogy here is the ‘nonlinear Schwarz's list’ [12]) and most exhaustive references lists along the lines.

In this work we show that correlation between algebraic Picard–Hitchin solutions to the \wp -form (2) and, on the other hand, uniformizing Fuchsian equations leads to the new and infinite families of explicitly parameterized algebraic curves together with that constitutes the base of analysis on Riemann surfaces: Abelian integrals and their differential calculus. It is worthy of special emphasis that while examples of uniformizing functions (in Klein’s terminology Hauptmoduln [28], i.e. principal moduli) are known, few as they are, the state of the art as to the integrals is still that no uniformizing τ -representation for a basis of Abelian integrals is known for any one \mathcal{R} of genus $g > 1$. However constructions of integrals will be postponed to a separate continuation of this work since they lead to many other interesting consequences.

The most remarkable feature of the proposed family is that it is infinite, uniformly describable and fundamental group representation of corresponding orbifolds (Riemann surfaces) coincides with the automorphism group of a function field generator, that is Painlevé function $y(\tau)$ itself. Moreover, the computational apparatus involves the *new* kind of theta-constants and we develop the differential calculus not only for meromorphic objects on our \mathcal{R} ’s (functions and differentials as theta-ratios) but for these everywhere holomorphic functions as well.

Picard–Hitchin’s class lies in the base of our constructions but extends further. Most nice examples we exhibit in the text are the hyperelliptic curves since they have numerous applications. Although explicitly describable curves appear abundantly, we do not touch on the classification problems; rather we stress the *ways of getting* the analytic results accompanying the uniformization theory because all the attendant problems of the theory have got an explicitly solvable form. The remarkable facet is that this is the first instance of that situation and it comes from the physical equation \mathcal{P}_6 ; this explains the presence of a large number of examples/exercises.

One further motivation that should be mentioned is that Painlevé equations have the extensive physical literature and, on the other hand, general theory of uniformization has long experienced the lack of nontrivial illustrative applications. Even computational problems in the theory are not trivial and recently this subject has received increased attention in the context of another (aside from integrals and functions) fundamental object: the Schottky–Klein prime function; see [20, 21] and references therein. Although these works are concerned with multiconnected versions of \mathcal{R} ’s, the main difficulties focus on computational and analytic aspects anyway¹. As for applications, it is worth noting that inversions of Fuchsian equations we consider give us the explicit and global representations for canonical maps between moduli spaces of one-parameter families of Calabi–Yau mirror pairs (X, X^*) (IIA/IIB string duality), which are better known as the famous mirror maps, realized through the Picard–Fuchs equations. See, e.g., [46, 64] for references source. On the other hand, universal orbifold Hauptmoduln (see Sect. 2.3 for explanations)

¹D. Crowdy notices properly in recent survey [22, p. T206]: ‘While many abstract theoretical results exist, the mathematical literature has been lacking in *constructive* techniques for solving such problems . . . , the mathematics applicable to such problems has widespread relevance to all sorts of other problems in mathematical physics, classical physics, integrable systems theory and the theory of ordinary and PDEs’.

parametrize *arbitrary \mathcal{R} 's of nonzero genera* and conversely, any Riemann surface (punctured or compact) is necessarily described by zero genus Hauptmoduln [14]. In particular, this means that there is actually no need to take, as the starting point, the hyperbolic Poincaré polygons of genera $g > 1$ because they correspond to Fuchsian equations with algebraic coefficients. Among other things, monodromies of such equations have, apart from non-commutativity, very complicated geometric structure.

1.2. Outline of the paper. The article is organized as listed in Contents and consists of three parts. The first one (sects. 2–6) deals with the uniformization of Picard–Hitchin's algebraic solutions. The second part (sects. 7–9) is devoted to consequences, relationships to other important equations, and examples. The third part (sects. 10–11) sums up the two previous ones from the general differential viewpoint.

In Sect. 2 we fix notation and briefly expound some background information for uniformization theory: Fuchsian equations, their monodromies, inversion problems, and transformations between equations. We also introduce the convenient object—the meromorphic \mathfrak{D} -derivative—and the notion of universal uniformization. At the heart of constructing the new uniformizable curves is a simple lemma on transformations (Lemma 1).

In Sect. 3 we recall the Painlevé substitution and inspect some little-known facts about Picard's solution.

In Sect. 4 we simplify the original parametric form of Hitchin's solution and present results in the language of uniformizing theory (Sect. 4.2): nontrivial Fuchsian equations and uniform representation for functions in terms of new kind θ -constants. These objects, along with the uniformizing functions and Abelian differentials, admit the effective and *closed* differential computations. These technicalities are expounded in Sect. 5.

Section 6 is devoted to transformation groups. We first tabulate the general transformation rules for θ -functions in Sect. 6.1 and then, on the basis of this, in Sect. 6.2 we describe uniformizing groups for Picard's curves.

Section 7 contains basically of examples. One exhibits ways of generation of uniformizable curves being no solutions to the \mathcal{P}_6 -equation; we call this a *tower of curves*. Among these are some non-hyperelliptic curves and three classical hyperelliptic examples (Sects. 7.1 and 7.2). We conclude this section with the general recipe of getting the formulae.

In Sect. 8 we show that Picard–Hitchin's solutions have direct links with many important Fuchsian equations. For example, a simple Picard's solution yields an equation from the famous Chudnovsky list of the four Heun equations (Sect. 8.1). We give some explanations as to how these equations can be related to each other. The same Picard's solution is related, through corresponding Fuchsian equation, to the famous Apéry linear ODE; we discuss this fact at greater length in Sect. 8.2. All these facts allow us to write down the explicit τ -representations for associated inversion problems. Although the subsequent section (Sect. 8.3) is devoted to further illustration of the ‘ θ -apparatus’ (including the nice hyperelliptic example $z^2 = x^6 - 1$) and relationships between Hauptmoduln and solutions of Fuchsian equations, the main purpose of this section is to anticipate an important generalization which will be expounded in Sect. 10.

It is notable that the Picard–Hitchin class has links with non Picard–Hitchin’s algebraic solutions to \mathcal{P}_6 . The latter curves also produce solvable Fuchsian equations. This is the subject matter of Sect. 9.

Section 10 is devoted to differential structures on \mathcal{R} ’s and, in particular, to systematization of results of sects. 5 and 8.3. We explain that, besides the functions and differentials, one should introduce the notion of analytic connection on \mathcal{R} . Sections 10.1 and 10.2 show how this object is described by means of certain ODEs through the unique fundamental scalar (automorphic function). We present the regular recipe of getting such ODEs and exhibit nontrivial examples and exercises.

In final section (Sect. 11) we sketch a relationship between the preceding material and transcendental (solvable) Fuchsian equations on tori. Construction of such equations results from the fact that the majority of our algebraic curves cover elliptic tori.

2. BACKGROUND MATERIAL

2.1. Uniformization, Schwarzsians, and Fuchsian equations. Classically, in the language of differential equations, uniformization of Riemann surfaces of finite genera is described by linear differential equations of Fuchsian class [27]. If $x = \chi(\tau)$ is a generator of the function field of meromorphic automorphic functions on a Riemann surface \mathcal{R} of some algebraic curve (3) then the global uniformizing parameter τ is determined as the quotient

$$\tau = \frac{\Psi_2(x)}{\Psi_1(x)} \quad (4)$$

of two linearly independent solutions to the certain Fuchsian equation of 2nd order [27, 44]

$$\Psi_{xx} = \frac{1}{2} \mathcal{Q}(x, y) \Psi. \quad (5)$$

Equation of the same form $\psi_{yy} = \frac{1}{2} \tilde{\mathcal{Q}}(x, y) \psi$, where $\psi = \sqrt{y_x} \Psi$, determines the τ through the second function $y = y(\tau)$. That equation (5) is of Fuchsian class implies that function $\mathcal{Q}(x, y)$ is bound to be rational. This function (or $\tilde{\mathcal{Q}}$) completely determines all the analysis on \mathcal{R} . The ratio τ itself, as a function of x , is a solution of nonlinear non-autonomous ODE of 3rd order [27]

$$\frac{\tau_{xxx}}{\tau_x} - \frac{3}{2} \frac{\tau_{xx}^2}{\tau_x^2} = -\mathcal{Q}(x, y), \quad (6)$$

better known as the Schwarz equation, and left hand side of this equation is traditionally designated as Schwarz’s derivative [59]: $\{\tau, x\} = -\mathcal{Q}(x, y)$.

Solutions of equations (5) and (6) are essentially multi-valued functions of the variable x . This multi-valuedness is described by a transformation group and the group itself is nothing but matrix (2×2) -representation of the monodromy group \mathfrak{G}_x of equation (5) [27]:

$$\mathfrak{G}_x : \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

This transformation entails the main property of function $x = \chi(\tau)$, namely, the property of being automorphic:

$$\chi\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(\tau) \Rightarrow \mathbf{Aut} \chi(\tau) =: \mathfrak{G}_x.$$

If this $\chi(\tau)$, as analytic function of τ , turns out to be globally single-valued in the domain of its existence² \mathbb{D} and Poincaré polygon [28, 44] for monodromy \mathfrak{G}_x has finite topological genus we may think of $\chi(\tau)$ as a finite order meromorphic function on factor $\mathbb{D}/\mathfrak{G}_x$ being generally an orbifold [65] and, upon compactification (if required), becoming a Riemann surface of some algebraic curve (3), possibly sphere $\mathbb{P}^1(\mathbb{C})$. It is known that this construction can always be realized by a Kleinian group with an additional condition that this group has an invariant circle and determines thereby a Fuchsian group of 1st kind [27, 44]. We normalize this circle to be the real axis \mathbb{R} and universal cover (where τ ‘lives’) to be the upper half plane $\mathbb{H}^+ \ni \tau$, that is $\Im(\tau) > 0$; the domain \mathbb{D} thus becomes \mathbb{H}^+ .

If Poincaré domain for the single-valued $\chi(\tau)$ has some punctures then compactification $\overline{\mathbb{H}^+/\mathfrak{G}_x}$ of this polygon is unique [8]. In other words, from the analytic view-point functions on that compactified objects are equally well as functions on pure hyperbolic Riemann surfaces without punctures. See Remarks 1 and 5 further below and Sect. 7.1 for additional explanations.

With the exception of accessory parameters problem the most important problems in the field are 1) explicit solutions to (5) and 2) explicit representation for inversions of the ratio (4). All the currently known solutions to the first of these problems are reduced to the hypergeometric functions and triangle groups [28]; though there are curves (Shimura curves) with their Fuchsian equations (5) having no such a type of reduction (see, e.g., [17, 43, 25] for explicit formulae). As for the second problem (the inversion problem) the number of solvable examples falls far short of the first one and is limited only by particular triangle groups, namely, groups commensurable with $\Gamma(1)$. Even the most famous curve $x^3y + y^3z + z^3x = 0$ [45] was uniformized by Klein not through its ‘native’ hyperbolic (2,3,7)-triangle group but through the theta-constants associated with modular group $\Gamma(7)$, i.e. matrices congruent to the identity mod 7 [45]. This point is a manifestation of the fact that no one representation for uniformizing function (to say nothing of Abelian integrals) associated with any non-modular Fuchsian equation is known hitherto.

2.2. Meromorphic derivative. Algebraic functions like x , y , etc are not only meromorphic single-valued objects on \mathcal{R} ’s. Complete analysis should necessarily include meromorphic Abelian differentials and their integrals as well. Rather than manipulate with non-autonomous equations (6) and multi-valued inversions of multi-valued objects, it is convenient to invert Schwarz’s derivative $\{\tau, x\}$ to the object $\{x, \tau\}/\dot{x}^2$ and to handle the

²The domain \mathbb{D} and property of being single-valued crucially depend on parameters of function \mathcal{Q} . Those parameters that do not affect the local analysis but determine the global domain and the global single-valuedness (therefore topology of \mathfrak{G}_x and genus) are called accessory parameters [28, 27, 65, 8]. We shall call these parameters *correct* if $x = \chi(\tau)$ is a single-valued function everywhere in \mathbb{D} .

meromorphic derivative \mathfrak{D} [16]:

$$\mathfrak{D} : \quad [x, \tau] := \frac{\ddot{x}}{\dot{x}^3} - \frac{3}{2} \frac{\ddot{x}^2}{\dot{x}^4},$$

where dots above the symbols, as always in the sequel, stand for derivatives with respect to τ . For the reasons above, there is no point in distinguishing Fuchsian linear or Schwarz's nonlinear equations

$$[x, \tau] = \mathcal{Q}(x, y). \quad (7)$$

Hence, in order to avoid lengthening terminology we shall use the standard notions—regular singularities, branch places, Fuchsian exponents, et cetera [27, 65]—for both kinds of equations. In particular, we shall use as synonyms the notions automorphisms **Aut**, monodromies, and groups like \mathfrak{G}_x and refer to linear equations (5) and their nonlinear Schwarz varieties (7) merely as Fuchsian equations. The famous theorem of Klein–Poincaré–Koebe [28, 8] guarantees the availability of what is called the unique Fuchsian monodromy of 1st kind for equation (7), that is uniqueness of function $\mathcal{Q}(x, y)$ with a given set of x -singularities $\{E_k\}$. Throughout the paper we consider as equivalents Fuchsian equations in the normal Klein form (5) and equations of the form

$$\psi_{xx} + p(x, y)\psi_x + q(x, y)\psi = 0 \quad (8)$$

because transition between these forms is achieved by the known linear transformation

$$\Psi = \psi \cdot \exp \frac{1}{2} \int p \, dx;$$

it has no effect on the ratio (4).

2.3. Universal uniformization and transformations between Fuchsian equations.

If the structure of \mathcal{Q} is such that in the neighborhood of some point $x = E$ we have

$$[x, \tau] = -\frac{1}{2} \frac{1}{(x - E)^2} + \dots,$$

then the local behavior of function $\chi(\tau)$ will be exponential [65, Ch. 5], [27]. This being so, arbitrary algebraic ramification $y \sim (x - E)^q + \dots$ ($q \in \mathbb{Q}$) is transformed into the locally single-valued dependence $y(\tau)$. This motivates the following definition.

Definition. *Universal uniformization through punctures. The meromorphic automorphic function $x = \chi(\tau)$ on \mathbb{H}^+ is said to be the universal uniformizing function for the set of points $\{E_k\}$ if it has the exponential behavior*

$$x = E + a \exp \left(\frac{-\pi i n}{\tau - \tau_0} \right) + \dots$$

in neighborhoods of E 's as $\tau \rightarrow \tau_0 + 0i$ and $\tau_0 \in \mathbb{R}$. If $\tau_0 = i\infty$ we define

$$x = E + a \exp(\pi i n \tau) + \dots$$

In other words, local monodromy \mathfrak{G}_x for universal uniformizing function is determined by the parabolic singularities E_k in (5) and implies the exponents above. The definition does not forbid to have non-parabolic, i.e. conical, singularities. It follows that *any* algebraic function of x , say (3), with arbitrary ramifications only at points $x = E_k$ becomes

a single-valued function of τ . Correspondence between the exponential behavior, that is puncture, and the \mathfrak{D} -object is perhaps most easily clarified by observing that meromorphic derivative of a function can be represented through the one of its logarithm:

$$[x, \tau] = \frac{1}{(x - e)^2} \left([\ln(x - e), \tau] - \frac{1}{2} \right).$$

The classical and simplest example is a uniformization of a 3-punctured sphere:

$$[x, \tau] = -\frac{1}{2} \frac{x^2 - x + 1}{x^2(x - 1)^2}. \quad (9)$$

Solution of this equation is the x -function in Painlevé substitution (formula (16)).

Remark 1. An interrelation needs to be understood between parametrizations, uniformization, and the curve itself. The curve is not invariant object since we may do birational transformations. It therefore has no punctures or conical singularities. The only invariant object is a Riemann surface \mathcal{R} with a pure hyperbolic system of generators representing its fundamental group $\pi_1(\mathcal{R})$. On the other hand, compactification is unique and complex analytic objects on \mathcal{R} —generators $x(\tau), y(\tau)$ of a function field and Poincaré domain \mathbb{D} —have been subordinated to the only fundamental analytic property: the global single-valuedness with due regard for corresponding factor topology. In this respect the uniformization with punctures is not inferior than pure hyperbolic one³. Moreover, punctures are not necessary conditions to construct universal uniformization since there exist ‘non-punctured’ universal ones. Every compact Riemann surface \mathcal{R} does indeed have a naturally associated orbifold \mathfrak{T} *without* punctures and pure hyperbolic representation of $\pi_1(\mathcal{R})$ turns out to be merely a subgroup of $\pi_1(\mathfrak{T}) = \mathfrak{G}_x$, where $\mathfrak{G}_x = \mathbf{Aut}(x)$, and function $x = \chi(\tau)$ is automorphic with respect to representation of this $\pi_1(\mathcal{R})$. Topological arguments show that group \mathfrak{G} uniformizing the curve (3) and representing $\pi_1(\mathcal{R})$ is to be taken as $\mathfrak{G} = \mathfrak{G}_x \cap \mathfrak{G}_y$. This is the subject matter of recent work [14]. It is of interest to remark here that Fuchsian equations for subgroups/curves have not got to the second (function) volume of the monumental Fricke–Klein treatise on automorphic functions [28], whereas Picard–Hitchin’s and many other modular equations provide examples of such constructions.

Meromorphic derivative \mathfrak{D} can be calculated for any object on \mathcal{R} . If this object is an element of function field then it is algebraically related to any other one. This algebraic relation, considered as a change of variable, transforms Fuchsian equations one into the other.

Lemma 1. *If $z = R(x)$ is any function of x then equation $[x, \tau] = \mathcal{Q}(x, y)$ implies that*

$$[z, \tau] = [R(x), x] + \frac{1}{R_x^2} \mathcal{Q}(x, y).$$

³Indeed, considering the object/sphere $x^2 + y^2 = 4$ we do not think of its two parametrizations $\{x = 2\sin\tau, y = 2\cos\tau\}$ and $\{x = \tau + \tau^{-1}, iy = \tau - \tau^{-1}\}$ as one is better and other is worse. It is clear that the obvious transformation $\tau \mapsto e^{i\tau}$ realizes a translation between these ‘punctured’ and ‘non-punctured’ uniformizations. The remarkable fact is that there exists an analog of that transition for higher genera and it is described by ODEs; the first explicit instance for the case $g = 2$ has been exhibited in work [16].

This is of course the \mathfrak{D} -version of the transformation law for Schwarz's derivative of a function composition $\tau \circ \mu$: if $\tau = f(\mu)$ and $\mu = g(z)$ then we have

$$\{\tau, \mu\} d\mu^2 + \{\mu, z\} dz^2 = \{\tau, z\} dz^2. \quad (10)$$

In spite of seemingly triviality, this lemma has a fundamental meaning because algebraic curves may form *towers* and integrability of their Fuchsian equations is in effect an integrability of a *single* equation (see Sect. 7). Global parameters τ 's for all of these equations/curves are the one common τ and we search for relations between these curves. In our case, these relations are algebraic/rational ones to Hauptmodul $x = \chi(\tau)$ and the problem consists in finding these substitutions and representations for various (Haupt) Moduln. In the following, we shall exhibit such results coming from the Picard–Hitchin solutions to \mathcal{P}_6 . Of course, our parametrizations correspond to finite covers of the punctured spheres and orbifolds since the full modular group $\Gamma(1) := \text{PSL}_2(\mathbb{Z})$ and its subgroups like $\Gamma(2)$ form presently the only class for which explicit inversions of (4) are known. However, as pointed out above, we get an extension of the classical family of Jacobi's ϑ -constants and Dedekind's eta-function.

2.4. Notation. Picard–Hitchin's solutions involve nontrivial combinations of Jacobi's and Weierstrass's functions and we use intensively many of their properties without explicit mentioning. Among enormous literature on this subject, in most of cases Schwarz's collection of Weierstrass's and Jacobi's classical results [63] is by no means lacking and the four volume set by Tannery & Molk [60], as a formulae source, hitherto contains most exhaustive information along these lines.

We use the four Jacobi's functions θ_k , introduced by Hermite as θ -functions with characteristics [35, p. 482], in the following definition [62]:

$$\theta_{[\beta]}^{[\alpha]}(z|\tau) = \sum_{k=-\infty}^{\infty} e^{\pi i (k+\frac{\alpha}{2})^2 \tau + 2\pi i (k+\frac{\alpha}{2})(z+\frac{\beta}{2})}. \quad (11)$$

Therefore, $\theta_1 = -\theta_{[1]}^{[1]}$, $\theta_2 = \theta_{[0]}^{[1]}$, $\theta_3 = \theta_{[0]}^{[0]}$, $\theta_4 = \theta_{[1]}^{[0]}$. Nullwerthe of θ 's, termed usually the ϑ -constants, are the values of $\theta(z|\tau)$ under $z = 0$, that is $\vartheta_k := \vartheta_k(\tau) = \theta_k(0|\tau)$. We introduce the fifth and independent object θ'_1 as a derivative of the θ_1 -series:

$$\theta'_1(z|\tau) = \pi e^{\frac{1}{4}\pi i \tau} \sum_{k=-\infty}^{\infty} (-1)^k (2k+1) e^{(k^2+k)\pi i \tau} e^{(2k+1)\pi i z}.$$

The standard Weierstrassian functions $\sigma, \zeta, \wp, \wp'(z|\omega, \omega')$ correspond to the set of half-periods (ω, ω') [63, 62, 60]. By virtue of homogeneous relations, say $\alpha^2 \wp(\alpha z|\alpha\omega, \alpha\omega') = \wp(z|\omega, \omega')$, we may always put any half-period to unity and, as throughout the paper, handle functions like $\wp(z|1, \tau) =: \wp(z|\tau)$, etc, where τ stands for the ratio ω'/ω . If $\Im(\tau) < 0$, then one interchanges ω and ω' . Period of canonical meromorphic elliptic integral ζ is

usually denoted as $\eta(\tau) := \zeta(1|\tau)$. Translation of Weierstrassian functions into the θ -functions is realized by means of the following formulae [63, 60]:

$$\begin{aligned}\wp(2z|\tau) &= \frac{\pi^2}{12} \left\{ \vartheta_3^4(\tau) + \vartheta_4^4(\tau) + 3\vartheta_3^2(\tau)\vartheta_4^2(\tau) \frac{\theta_2^2(z|\tau)}{\theta_1^2(z|\tau)} \right\}, \\ \zeta(2z|\tau) &= 2\eta(\tau)z + \frac{1}{2} \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}, \quad \wp'(2z|\tau) = -\pi^3 \hat{\eta}^9(\tau) \frac{\theta_1(2z|\tau)}{\theta_1^4(z|\tau)},\end{aligned}\tag{12}$$

where $\hat{\eta}(\tau)$ is the function of Dedekind:

$$\hat{\eta}(\tau) = e^{\frac{\pi i}{12}\tau} \prod_{k=1}^{\infty} (1 - e^{2k\pi i\tau}) = e^{\frac{\pi i}{12}\tau} \sum_{k=-\infty}^{\infty} (-1)^k e^{(3k^2+k)\pi i\tau}.$$

It is differentially and algebraically related to the η, ϑ -constants:

$$\frac{1}{\hat{\eta}} \frac{d\hat{\eta}}{d\tau} = \frac{i}{\pi} \eta, \quad 2\hat{\eta}^3(\tau) = \vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau).\tag{13}$$

3. PICARD'S SOLUTION, REVISITED

Contemporary mentions of this solution refer usually to (1) however ODE derived by Picard himself differs from Painlevé form (1). As said above, Picard was not concerned with equation (1) or its particular case. At the end of his mémoire [56] he offered as an example the une équation différentielle curieuse with fixed critical points satisfied by Jacobi's elliptic sinus $\text{sn}(a\omega + b\omega'; k)$ considered as function of Legendre's modulus $k^2 = x$. Picard denoted this function by $u(x)$ and deduced the equation

$$\begin{aligned}\frac{d^2u}{dx^2} - \left(\frac{du}{dx} \right)^2 \frac{u(2xu^2 - 1 - x)}{(1 - u^2)(1 - xu^2)} \\ + \frac{du}{dx} \left[\frac{u^2 - 1}{(1 - x)(1 - xu^2)} + \frac{1}{x} \right] - \frac{1}{4} \frac{u(1 - u^2)}{x(1 - x)(1 - xu^2)} = 0\end{aligned}\tag{14}$$

(in original paper [56] on p. 298 the multiplier $\frac{1}{4}$ in front of last term was missing). Transformation between Picard's and Painlevé equations (2), (14) can be derived from the substitution of Painlevé $(x, y) \mapsto (z, \tau)$ turning equation (1) into (2). Indeed, converting Painlevé formulae [54, p. 1117] into the theta-functions, we get the substitution [4]

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \quad y = -\frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)} \frac{\theta_2^2(\frac{1}{2}z|\tau)}{\theta_1^2(\frac{1}{2}z|\tau)}\tag{15}$$

and its (ϑ, \wp) -equivalent [51]

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \quad y = \frac{1}{3} + \frac{1}{3} \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)} - \frac{4}{\pi^2} \frac{\wp(z|\tau)}{\vartheta_3^4(\tau)}.\tag{16}$$

Hence relation between Picard's u and Painlevé y is nothing but the known relation between functions \wp and sn :

$$\wp(z + \tau|\tau) - e'(\tau) = \frac{\pi^2}{4} \vartheta_2^4(\tau) \cdot \text{sn}^2\left(\frac{\pi}{2} \vartheta_3^2(\tau) z; k\right)$$

and therefore

$$u\left(\frac{1}{x}\right) = \sqrt{y(x)}. \quad (17)$$

The first equation in (16) is invertible and inversion itself involves Legendre's objects K, K' [60]:

$$\tau = i \frac{K(\sqrt{x})}{K'(\sqrt{x})}. \quad (18)$$

The quantities K and K' can be considered as hypergeometric series

$$K(\sqrt{x}) = \frac{\pi}{2} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \middle| x\right), \quad K'(\sqrt{x}) = \frac{\pi}{2} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \middle| 1-x\right) \quad (19)$$

or, depending on preference, as complete elliptic integrals [2]

$$K(\sqrt{x}) = \int_0^1 \frac{d\lambda}{\sqrt{(1-\lambda^2)(1-x\lambda^2)}}, \quad K'(\sqrt{x}) = \int_0^1 \frac{d\lambda}{\sqrt{(\lambda^2-x)(1-\lambda^2)}}.$$

The relation (17) leads to the following property of Picard's case.

Proposition 1. *Picard's solution of equation (1) is a perfect square:*

$$y_{\text{Pic}} = -\sqrt{x} \frac{\theta_2^2 \left(A \frac{K(\sqrt{x})}{K'(\sqrt{x})} + B \middle| \frac{iK(\sqrt{x})}{K'(\sqrt{x})} \right)}{\theta_1^2 \left(A \frac{K(\sqrt{x})}{K'(\sqrt{x})} + B \middle| \frac{iK(\sqrt{x})}{K'(\sqrt{x})} \right)}, \quad (20)$$

where A, B are free constants.

This is a special feature of Picard's parameters since square root of arbitrary solution to equation (1) contains no movable ramifications and satisfies another equation with the Painlevé property, that is equation (14). In this regard Hitchin's solution is much more nontrivial (see Sect. 4). The \wp -form of Picard's solution was considered comprehensively by Fuchs [30] and reinspected by Mazzocco in [52].

On the other hand, Gambier's list of fifty transcedents [31] is complete and, contrary to (17), there is bound to be a change of independent variable $x \mapsto z(x)$ transforming (14) into the one of equations from this list. It is not difficult to see that it can be only Picard's case of equation (1). See the number (49) on p. 17 in [31] and, concerning the change itself, the case III on p. 323 in [39]; the latter transforms into the case viii on p. 326 in the same place. Carrying out calculations and gathering intermediate substitutions we get the following symmetry.

Proposition 2. *Let $\alpha = \beta = \gamma = \delta = 0$. Then the transformation*

$$y(x) \mapsto \left\{ \frac{y \left(\left(\frac{\sqrt{x}-1}{\sqrt{x}+1} \right)^2 \right) + \frac{\sqrt{x}-1}{\sqrt{x}+1}}{y \left(\left(\frac{\sqrt{x}-1}{\sqrt{x}+1} \right)^2 \right) - \frac{\sqrt{x}-1}{\sqrt{x}+1}} \right\}^2$$

is a function automorphism of equation (1) preserving the perfect square.

Transformations of such a kind are the subject of a more general theory of quadratic transformations and additional details on this theory can be found in work [61].

4. UNIFORMIZATION OF PICARD–HITCHIN’S CURVES

4.1. Hitchin’s solution. General group of invariance of the Picard–Painlevé property is a birational group [18]. This transformation group includes derivatives (see Okamoto’s transformations (26) further below) and leads to Hitchin’s solution which has hitherto remained most nontrivial and ‘rich’ solution of all those currently known. Its parametric form is as follows [36, Theorem 6]:

$$\wp(z|\tau) = \wp(A\tau + B|\tau) + \frac{1}{2} \frac{\wp'(A\tau + B|\tau)}{\zeta(A\tau + B|\tau) - A\eta'(\tau) - B\eta(\tau)}. \quad (21)$$

with the same meaning of A , B , and τ as in Proposition 1. In implicit form this solution was also obtained by Okamoto [53, p. 366]. In addition to (21), Hitchin suggested a theta-function form for his solution which turned out to be rather complicated since it contains the set of functions $\vartheta, \vartheta', \vartheta''', \theta, \theta', \theta'', \theta'''$. Reproduce the solution in Hitchin’s original notation on p. 33 of [36]:

$$\begin{aligned} y(x) &= \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3} \left(1 + \frac{\vartheta_3^4(0)}{\vartheta_4^4(0)} \right) \\ &+ \frac{\vartheta_1'''(\nu)\vartheta_1(\nu) - 2\vartheta_1''(\nu)\vartheta_1'(\nu) + 4\pi i c_1 (\vartheta_1''(\nu)\vartheta_1(\nu) - \vartheta_1'^2(\nu))}{2\pi^2\vartheta_4^4(0)\vartheta_1(\nu)(\vartheta_1'(\nu) + 2\pi i c_1 \vartheta_1(\nu))}, \end{aligned} \quad (22)$$

where $\nu = c_1\tau + c_2$. Below is simplification of Hitchin’s formulae resulting in a full parametrization of subsequent curves.

Theorem 1. *General solution to Hitchin’s case of equation (1) has the form*

$$y = \frac{\sqrt{x}}{\theta_1^2} \left\{ \frac{\pi\vartheta_2^2 \cdot \theta_2 \theta_3 \theta_4}{\theta_1' + 2\pi A \theta_1} - \theta_2^2 \right\}, \quad (23)$$

where functions θ_1', θ_k are understood to be equal to $\theta_1', \theta_k \left(A \frac{K(\sqrt{x})}{K'(\sqrt{x})} + B \left| \frac{iK(\sqrt{x})}{K'(\sqrt{x})} \right. \right)$ and $\vartheta_2 = \vartheta_2 \left(\frac{iK(\sqrt{x})}{K'(\sqrt{x})} \right)$.

Proof. Meromorphic functions on Riemann surfaces are expressed through ratios of theta-functions and therefore the set of θ -functions in (22) is excessive. Based on conversion formulae (12) and duplication rules for θ -functions [60] we obtain that (21) \rightsquigarrow (23). First formula in (12) explains also the transition between (15) and (16). \blacksquare

Remark 2. An important point here is the fact that function θ'_1 should play an independent part in the θ -calculus along with the four functions θ_k . An explanation of this role of θ'_1 is as follows. Weierstrass' ζ -function is the canonical meromorphic integral on elliptic curves. It never reduces to holomorphic or logarithmic integrals, or meromorphic elliptic functions since it is related to them only through a derivative. On the other hand $\zeta \sim \theta'_1/\theta_1$.

4.2. Uniformization of Hitchin's curves. Algebraic solutions (3) of Picard's class correspond to constants [56, 30, 52]

$$A\tau + B = \frac{\nu}{2N}\tau + \frac{\mu}{2N}, \quad N \in \mathbb{Z} \setminus \{0\}. \quad (24)$$

By virtue of Theorem 1 algebraic solutions to the Hitchin case are effectively described as well as Picard's class.

Theorem 2. *Let $\tau \in \mathbb{H}^+$, $\vartheta_k = \vartheta_k(\tau)$, and the functions $\theta'_1, \theta_k(A\tau + B|\tau)$ be taken with arguments (24). Then algebraic solutions (3) to the Hitchin case of parameters in (1) have the following parametrization by single-valued functions:*

$$x = \frac{\vartheta_4^4}{\vartheta_3^4}, \quad y = \frac{\vartheta_4^2 \theta_2}{\vartheta_3^2 \theta_1^2} \left\{ \frac{\pi \vartheta_2^2 \cdot \theta_3 \theta_4}{\theta'_1 + \pi i \frac{\nu}{N} \theta_1} - \theta_2 \right\}. \quad (25)$$

Proof. That functions (25) satisfy some algebraic dependence (3) will be apparent from the fact that (24) generates an algebraic family of Picard P (20) and the latter is related to Hitchin's solutions H through the Okamoto transformations [53, 33]:

$$H = P + \frac{P(P-1)(P-x)}{x(x-1)P_x - P^2 + P}, \quad P = H - \frac{H(H-1)(H-x)}{x(x-1)H_x + \frac{1}{2}H^2 - xH + \frac{1}{2}x}. \quad (26)$$

These maps transform algebraic dependencies into the same ones. \blacksquare

Corollary 1. *Let us fix N in (24). Then every Picard's solution is a rational function $P = R(x, H)$ on Hitchin's curve $F_N(x, H) = 0$ and conversely. Picard's and Hitchin's solutions, along with any algebraic solutions tied by a certain Okamoto transformation, define isomorphic curves.*

Example 1. Under $N = 3$ and $(\nu, \mu) = (0, 1)$ the birational isomorphism read as follows

$$P = 1 - \frac{(x-1)^2}{(H-1)^2}, \quad H = \frac{3}{2} \frac{x}{P} - \frac{1}{2}P.$$

In work [14] we conjectured that theta-constants of the general form $\theta(u(\tau)|\tau)$ can be relevant in uniformization theory. As we have seen now this is so indeed and the new family of theta-constants $\theta\left(\frac{\nu}{N}\tau + \frac{\mu}{N}|\tau\right)$ comes into play. The distinctive property of this family, as compared with classical modular $\hat{\eta}$, ϑ -constants [62], is the availability of the special constant $\theta'_1\left(\frac{\nu}{N}\tau + \frac{\mu}{N}|\tau\right)$ and its role in differential closure. One more feature is the general formula for all the family of uniformizing functions, which cannot be said of many subfamilies of modular equations. For example, algebraic dependencies between roots of Legendre's modulus $k^2(\tau)$ have the form $F(\sqrt[n]{k}(p\tau), \sqrt[m]{k}(s\tau)) = 0$, which is easily written down for any integer n, m, p, s , but uniformizing functions are known explicitly

only in cases $n, m = \{1, 2, 4\}$ (see the last paragraph in Sect. 14.6.3 of [26]). Among other things, the size of coefficients in modular equations, as is well known, rapidly growths as level increases [62]. We shall use the term θ -constant (θ' -constant) if z -arguments of the θ, θ' -functions (11) are some functions of the modulus τ .

Theorem 3. *Let $y(\tau)$ be a uniformizing representation (16) of the arbitrary algebraic Painlevé solution (3). Then $y(\tau)$ is single-valued and satisfies the Fuchsian ODE:*

$$[y, \tau] = ([F, x] - [F, y]) F_y^2 + 3 \frac{F_y}{F_x} \left(\ln \frac{F_y}{F_x} \right)_{xy} - \frac{1}{2} \frac{x^2 - x + 1}{x^2(x-1)^2} \frac{F_y^2}{F_x^2}, \quad (27)$$

where $[F, x], [F, y]$ are computed as the partial \mathfrak{D} -derivatives.

Proof. Algebraic dependence (3) and Lemma 1 imply that Fuchsian equations (7) for automorphic functions $x(\tau)$ and $y(\tau)$ are not independent. The function $x(\tau)$ is an elliptic modulus $k'^2(\tau)$; therefore it satisfies equation (9) and is a universal uniformizing function for the three points $x = \{0, 1, \infty\}$. Since Painlevé functions ramify only over these points, $y(\tau)$ is single-valued. Applying Lemma 1 and substitution (3) to equation (9), we get, after some simplification, the computational rule (27). ■

In what follows we shall show that equations (27), being nontrivial Fuchsian equations with algebraic coefficients, are integrable and have computable monodromies.

In many respects, the simplicity of functions (25) leads to that we have actually no need for equation of the curve. In addition to Puiseux series, differentiations, plotting of graphs, etc, this is especially true in applications wherein parametric representation is the best suited form for implicit solutions.

5. DIFFERENTIAL CALCULUS ON PICARD–HITCHIN CURVES

Complete analysis of meromorphic functions on Riemann surfaces must contain Abelian differentials but Theorems 2–3 do not touch on these objects. Picard–Hitchin’s curves (abbreviated further to PH-curves) contain ϑ -constants and some of their differential properties (not all) are known [62]. The following result has a technical characterization but is needed for completeness of differential computations with theta-functions.

Lemma 2 [14]. *Jacobi’s ϑ -constants form a differential ring over \mathbb{C} upon adjoining the period of meromorphic elliptic integral, that is Weierstrass’ η :*

$$\begin{aligned} \frac{d\vartheta_2}{d\tau} &= \frac{i}{\pi} \left\{ \eta + \frac{\pi^2}{12} (\vartheta_3^4 + \vartheta_4^4) \right\} \vartheta_2, & \frac{d\vartheta_4}{d\tau} &= \frac{i}{\pi} \left\{ \eta - \frac{\pi^2}{12} (\vartheta_2^4 + \vartheta_3^4) \right\} \vartheta_4, \\ \frac{d\vartheta_3}{d\tau} &= \frac{i}{\pi} \left\{ \eta + \frac{\pi^2}{12} (\vartheta_2^4 - \vartheta_4^4) \right\} \vartheta_3, & \frac{d\eta}{d\tau} &= \frac{i}{\pi} \left\{ 2\eta^2 - \frac{\pi^4}{12^2} (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8) \right\}. \end{aligned} \quad (28)$$

In order to obtain the general Abelian differential $R(x, y)dx$ on a PH-curve it is convenient to have a representation for basic differentials $dy_{\text{PIC}} = \dot{y}_{\text{PIC}} d\tau$ and $dy_{\text{HIT}} = \dot{y}_{\text{HIT}} d\tau$ independently of form of solution (3).

Theorem 4. *Let parameters A, B defined by (24) determine the algebraic solutions (20) and (25). Then PH-differentials $dy = \dot{y} d\tau$, as differentials on corresponding curves*

(3), are given by the following expressions:

$$\begin{aligned} \dot{y} &= \pi i \vartheta_3^6 \theta_1^2 \frac{y(y-1)(y-x)}{\vartheta_3^2 \theta_1^2 y + \vartheta_4^2 \theta_2^2} + \frac{\pi}{2i} \vartheta_3^4 (y-x)^2 + \frac{\pi}{2i} \vartheta_2^4 x && \text{(Hitchin's curves),} \\ \dot{y} &= i \vartheta_2^2 \left(\theta_1' + \pi i \frac{\nu}{N} \theta_1 \right) \frac{\theta_3 \theta_4}{\theta_2 \theta_1^2} y + \pi i \vartheta_3^4 y(y-1) && \text{(Picard's curves).} \end{aligned} \quad (29)$$

Proof. This is in fact the Okamoto transformations (26) resolved with respect to y_x which is proportional to \dot{y} . At first glance Corollary 1 contradicts to parametrization (25); whence it follows that function θ_1' must be a rational function of x, y . With use of theorems of addition/multiplication for ζ and the fact that $\zeta \sim \theta_1'/\theta$ we obtain that $\theta_1'(\frac{\nu}{N}\tau + \frac{\mu}{N}|\tau)$ is indeed the rational function of θ 's⁴ over $\mathbb{C}(\vartheta)$. In other words, Okamoto's transformations generate through the object θ_1' the basic Abelian differentials $dy(\tau)$ independently of N . Differentials (29) can also be considered as a θ -function τ -version of the transformations themselves. When simplifying (29) we used the equality $\dot{x} = -\pi i x \vartheta_2^4$ being a consequence of Lemma 2. \blacksquare

Remark 3 (exercise). Consider (29) as a Riccati equation $\dot{y} = f(\tau)y^2 + g(\tau)y + h(\tau)$ and derive a family of linear ‘integrable’ equations of the form $\Psi_{\tau\tau} = U(\tau)\Psi$.

Previous results were obtained with use of some particular properties of (1), (25)–(26), and curves; instead, we can use at once the differential properties of θ -functions involved in uniformizations (25) themselves. Below is a complete description of such properties. It will follow that differential analysis on PH-curves is a calculus of the ϑ - and θ -constants.

Theorem 5. *Let A, B be arbitrary quantities independent of τ . Then the set of theta-constants $\theta_1', \theta_k(A\tau + B|\tau)$ is differentially closed over $\mathbb{C}(\vartheta^2, \eta)$:*

$$\begin{aligned} \frac{d\theta_k}{d\tau} &= \frac{-i}{4\pi} (\theta_1' + 4\pi i A \theta_1) \theta_1' \frac{\theta_k}{\theta_1^2} + \frac{i}{2} \vartheta_k^2 \cdot (\theta_1' + 2\pi i A \theta_1) \frac{\theta_\nu \theta_\mu}{\theta_1^2} \\ &\quad + \frac{\pi i}{4} \left\{ \vartheta_3^2 \vartheta_4^2 \cdot \theta_2^2 - \vartheta_k^2 \vartheta_\mu^2 \cdot \theta_\nu^2 - \vartheta_k^2 \vartheta_\nu^2 \cdot \theta_\mu^2 \right\} \frac{\theta_k}{\theta_1^2} + \frac{i}{\pi} \left\{ \eta + \frac{\pi^2}{12} (\vartheta_3^4 + \vartheta_4^4) \right\} \cdot \theta_k, \\ \frac{d\theta_1'}{d\tau} &= \frac{i}{\pi} (3\theta_1' + 4\pi i A \theta_1) \left\{ \frac{\pi^2}{4} \vartheta_3^2 \vartheta_4^2 \cdot \frac{\theta_2^2}{\theta_1^2} + \eta + \frac{\pi^2}{12} (\vartheta_3^4 + \vartheta_4^4) \right\} \\ &\quad - \frac{i}{4\pi} (\theta_1' + 4\pi i A \theta_1) \frac{\theta_1'^2}{\theta_1^2} + \frac{\pi^2}{2i} \vartheta_2^2 \vartheta_3^2 \vartheta_4^2 \cdot \frac{\theta_2 \theta_3 \theta_4}{\theta_1^2}, \end{aligned} \quad (30)$$

where $k = 1, 2, 3, 4$ and ν, μ stand for

$$\nu = \frac{8k-28}{3k-10}, \quad \mu = \frac{10k-28}{3k-8}. \quad (31)$$

Proof. Denote by prime ' the derivative with respect to z . Then the logarithmic derivatives $\ln' \theta_k(z|\tau)$ satisfy identities which are sometimes present in the old literature [6, 60, 62]. In order to write them we observe that transformations (31) have third order

⁴An example: $\theta_1'(\frac{1}{4}) = \frac{\pi}{2}(\vartheta_3^2 + \vartheta_4^2)\theta_1(\frac{1}{4})$.

and θ -indices (k, ν, μ) run over cyclic permutations of the series $(2, 3, 4)$ when $k = 2, 3, 4$. The identities then can be written as follows

$$\frac{\theta'_k}{\theta_k} - \frac{\theta'_1}{\theta_1} = -\pi \vartheta_k^2 \cdot \frac{\theta_\nu \theta_\mu}{\theta_k \theta_1}.$$

This gives the derivatives $\theta'_{2,3,4}$ in terms of θ , θ'_1 . If $k = 1$ then these formulae work well since $\vartheta_k = \vartheta_1 \equiv 0$ and the terms $\sim \vartheta_k \theta_\nu$ drop out. Converting the identity $(\sigma')' = (\sigma')^2/\sigma - \sigma \wp$ into θ -functions, we obtain the expression for $(\theta'_1)'$ through θ and θ'_1 itself:

$$\frac{\partial \theta'_1}{\partial z} = \frac{\theta'_1^2}{\theta_1} - \pi^2 \vartheta_3^2 \vartheta_4^2 \cdot \frac{\theta_2^2}{\theta_1} - 4 \left\{ \eta + \frac{\pi^2}{12} (\vartheta_3^4 + \vartheta_4^4) \right\} \cdot \theta_1.$$

Invoking the heat equations $4\pi i \theta_\tau = \theta_{zz}$, $4\pi i \theta'_\tau = \theta'_{zz}$ and making the change $z \mapsto A\tau + B$, we arrive, upon simplification, at equations (30). Lemma 2 provides their differential closure and serves also the case $A\tau + B = 0$ as a limiting case of Eqs. (30). \blacksquare

The functions $\theta_k(A\tau + B|\tau)$ are the continual generalizations of θ 's with discrete characteristics and this theorem is a particular case of more general differential properties of θ -functions which are considered, in a context of the τ -function integrability of \mathcal{P}_6 , in work [15]. An addition of fourth equation in (28) is an important point. In a particular case, when A and B are chosen to be (24), we could formally get by without second equation in (30) but form of the first one would then depend on N . In other words, the Okamoto transformations (26), Abelian differentials (29), curves themselves (3), and Fuchsian equations (27), being rewritten in the language of θ -functions, constitute numerous and rather sophisticated η, ϑ, θ -identities and their differential consequences.

Remark 4. It is a good exercise to check the original solution of Hitchin (22) with use of Theorem 5 and Lemma 2.

6. GROUP TRANSFORMATIONS

6.1. Basic θ -transformations. In order to obtain the group properties of the automorphic functions under consideration we need transformation properties of their ingredients, i.e. θ, θ' -constants. The transformation for function θ_1 is known [60, 62] and usually written in the form

$$\theta_1\left(\frac{z}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d}\right) = \aleph \cdot \sqrt{c\tau + d} e^{\frac{\pi i c z^2}{c\tau + d}} \theta_1(z|\tau), \quad (32)$$

where \aleph denotes some eighth root of unity and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. However we shall require more detailed information. Let $[p]$ stand for integer part of the number p and suppose that c is normalized to be positive: $c > 0$.

Theorem 6. *The $\Gamma(1)$ -transformation law of the general θ -function. Let $\theta_{[\beta]}^{[\alpha]}$ be the theta-series with integer characteristics (11) and $n \in \mathbb{Z}$. Then*

$$\begin{aligned} \theta_{[\beta]}^{[\alpha-1]}(z|\tau + n) &= i^{\frac{n}{2}(1-\alpha^2)} \cdot \theta_{[\beta+n\alpha]}^{[\alpha-1]}(z|\tau), \\ \theta_{[\beta'-1]}^{[\alpha'-1]} \left(\frac{z}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d} \right) &= \mathfrak{E}_{\alpha\beta} \aleph \cdot \sqrt{c\tau + d} e^{\frac{\pi i c z^2}{c\tau + d}} \theta_{[\beta-1]}^{[\alpha-1]}(z|\tau), \end{aligned} \quad (33)$$

where multipliers \aleph and $\mathfrak{E}_{\alpha\beta}$ read

$$\mathfrak{E}_{\alpha\beta} = \exp \frac{\pi i}{4} \left\{ 2\alpha(\beta bc - d + 1) - \beta c(\beta a - 2) - \alpha^2 db \right\},$$

$$\aleph = \exp 3\pi i \left\{ \frac{a-d}{12c} - \frac{d}{6}(2c-3) + \frac{c-1}{4} \text{sign}(d) - \frac{1}{4} + \frac{1}{c} \cdot \sum_{[c/d]+1}^{c-1} \left[\frac{d}{c} k \right] k \right\} \quad (34)$$

and characteristics (α, β) , (α', β') are related through the linear transformations

$$\begin{aligned} \alpha' &= d\alpha - c\beta, & \alpha &= a\alpha' + c\beta', \\ \beta' &= -b\alpha + a\beta, & \beta &= b\alpha' + d\beta'. \end{aligned} \quad (35)$$

If $(\alpha, \beta) = (0, 0)$ the formula (33) turns into (32) so that θ_1 transforms into itself and $\theta_{2,3,4}$ are permuted one to another. Characteristics, as appeared in formula (33), have been chosen in order that the formula be most symmetric. Hermite represented \aleph through the sum of quadratic Gaussian exponents (it is known that these sums are not easily computed) and residue symbol [35, I: p. 485], whereas (33) contains one exponent of a rational number. Somewhat surprising facet is the fact that no such an explicit form of θ -transformation law seems to have hitherto been presented in the literature.

The multiplier \aleph is a common quantity to all the θ 's since it does not depend on characteristics (α, β) , whereas \mathfrak{E} does. Classical uniformizing functions are determined by classical ϑ -constants and the object $\mathfrak{E}_{\alpha\beta}(a, b, c, d)$. In turn, the well-known property of θ -characteristics

$$\theta_{[\beta+2n]}^{[\alpha+2m]}(z|\tau) = (-1)^{n\alpha} \theta_{[\beta]}^{[\alpha]}(z|\tau) \quad (36)$$

define congruence properties of the monodromies. Amongst the curves which have appeared, the three such functions encounter, each is universal uniformizing one:⁵

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \quad u = \frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)}, \quad v = \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}. \quad (37)$$

We have, however, (24) and therefore we should obtain automorphy factors for this kind θ, θ' -constants. If one considers Hitchin's algebraic solutions we should use the transformation law for θ'_1 which is derived from (32):

$$\theta'_1 \left(\frac{z}{c\tau+d} \middle| \frac{a\tau+b}{c\tau+d} \right) = \aleph \cdot \sqrt{c\tau+d} e^{\frac{\pi i c z^2}{c\tau+d}} \left\{ (c\tau+d) \theta'_1(z|\tau) + 2\pi i c z \theta_1(z|\tau) \right\}.$$

⁵In particular, third of these functions uniformizes the simplest spectral curve $w^3 = v^5 - v$ of a tetrahedrally symmetric non-singular monopole of charge 3 [38, Theorem 1]. See [14, p.258] for parametrization and [13] for special discussion of this monopole.

From the transformations above it follows immediately that the set of ϑ, θ -constants (24) is closed with respect to $\Gamma(1)$ under fix N :

$$\begin{aligned} \theta_{\alpha\beta}\left(\frac{\nu}{N}\tau + \frac{\mu}{N} \middle| \tau\right) &\xrightarrow{\widehat{\Gamma(1)(\tau)}} \theta_{\alpha\beta}\left(\frac{\nu}{N}\frac{a\tau + b}{c\tau + d} + \frac{\mu}{N} \middle| \frac{a\tau + b}{c\tau + d}\right) \\ &\equiv \theta_{\alpha'\beta'}\left(\frac{\nu a + \mu c}{N}\tau + \frac{\nu b + \mu d}{N} \middle| \tau\right) = \theta_{\alpha'\beta'}\left(\frac{\nu'}{N}\tau + \frac{\mu'}{N} \middle| \tau\right), \end{aligned} \quad (38)$$

so we can build transformations for θ -quotients.

In order to obtain the congruence for automorphism group of a function we should require that ϑ, θ -ratio defining the function transforms into itself; to do this, formulae (33), (35), (36), and multiplier \mathbf{E} provide all the required information. Furthermore, all the automorphic functions arisen from PH-curves are algebraically related to the function $x(\tau)$. Hence their automorphisms are commensurable with $\Gamma(2)$ in $\Gamma(1)$. This simplifies the analysis because transformation (33), when

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2n+1 & 2m \\ 2p & 2q+1 \end{pmatrix} \in \Gamma(2), \quad n, m, p, q \in \mathbb{Z} \quad (39)$$

splits into the separate formulae for all the θ_k .

Corollary 2. *The $\Gamma(2)$ transformation group is necessary and sufficient for each function θ_k to transform into itself. Under notation (39) we have:*

$$\begin{aligned} \theta_1\left(\frac{z}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d}\right) &= \mathbf{N} \cdot \sqrt{c\tau + d} e^{\frac{\pi i c z^2}{c\tau + d}} \theta_1(z|\tau), \\ \theta_2\left(\frac{z}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d}\right) &= i^{2q(p-1)+p} \mathbf{N} \cdot \sqrt{c\tau + d} e^{\frac{\pi i c z^2}{c\tau + d}} \theta_2(z|\tau), \\ \theta_3\left(\frac{z}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d}\right) &= i^{2q(p+1)-m(2n+1)+p} \mathbf{N} \cdot \sqrt{c\tau + d} e^{\frac{\pi i c z^2}{c\tau + d}} \theta_3(z|\tau), \\ \theta_4\left(\frac{z}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d}\right) &= i^{2n(m-1)-m} \mathbf{N} \cdot \sqrt{c\tau + d} e^{\frac{\pi i c z^2}{c\tau + d}} \theta_4(z|\tau), \end{aligned}$$

where \mathbf{N} is recomputed according to (34) and (39).

Proof. Analyzing (35) and (33) we obtain transformations for each of the θ -characteristics:

$$\theta_2 \mapsto \theta_{[d-1]}^{[c-1]}, \quad \theta_3 \mapsto \theta_{[b+d-1]}^{[a+c-1]}, \quad \theta_4 \mapsto \theta_{[b-1]}^{[a-1]}.$$

Requiring $\theta_{[d-1]}^{[c-1]} \simeq \theta_{[0]}^{[1]} = \theta_2$, $\theta_{[b+d-1]}^{[a+c-1]} \simeq \theta_{[0]}^{[0]} = \theta_3$, $\theta_{[b-1]}^{[a-1]} \simeq \theta_{[1]}^{[0]} = \theta_4$, we get the *linear* equations for a, b, c, d . Using (36), one obtains (39) and the desired transformation rules. \blacksquare

6.2. Congruences for Picard's groups. The formulae above, supplemented with the rule (36), allow us to derive the congruence properties of uniformizing functions under question. We shall restrict our consideration to Picard's curves.

Theorem 7. *The automorphism group \mathbf{G}_y of Picard's uniformizing function*

$$y(\tau) = -\frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)} \frac{\theta_2^2\left(\frac{\nu\tau+\mu}{2N} \middle| \tau\right)}{\theta_1^2\left(\frac{\nu\tau+\mu}{2N} \middle| \tau\right)} \quad (40)$$

coincides with the group $\mathfrak{G}_{\text{PIC}}^{(N)}$ uniformizing the Picard curve and is a free congruence subgroup of $\Gamma(2)$ with one of the following representations:

$$\mathfrak{G}_{\text{PIC}}^{(N)} : \quad \mathfrak{G} = \begin{pmatrix} 2Nn+1 & 2m \\ 2Np & 2Nq+1 \end{pmatrix}, \quad \begin{pmatrix} 2Nn+1 & 2Np \\ 2m & 2Nq+1 \end{pmatrix} = \mathfrak{G}^\top, \quad (41)$$

$$2mp - 2Nqn = q + n, \quad n, m, p, q \in \mathbb{Z}, \quad (42)$$

where $(\nu, \mu, N) \neq (0, \pm 1, \pm 2)$ and $N \leq N$ is computed through (ν, μ, N) . The group $\mathfrak{G}_{\text{PIC}}^{(N)}$ has the same topological genus as genus of Picard's curve $F_{(N)}(x, y) = 0$ under $(\nu, \mu) = (0, 1)$. Rank and generators of the group $\mathfrak{G}_{\text{PIC}}^{(N)}$ are computable.

Proof. The case $N = 1$ is trivial and we assume that $N > 1$. Since ϑ - and θ -constants transform separately into themselves, we obtain, based on Corollary 2 and invariancy condition for the theta-ratios in (40), that group \mathfrak{G}_y is not only commensurable with $\Gamma(2)$ but is its subgroup: $\mathfrak{G}_y \in \Gamma(2)$. We know also that $\Gamma(2) = \mathfrak{G}_x$ and (x, y) are generators of the function field on the curve $F_{(N)}(x, y) = 0$; hence it follows that the image of this curve in \mathbb{H}^+ coincides with geometrical polygon for the group $\mathfrak{G}_{\text{PIC}}^{(N)}$. Therefore $\mathfrak{G}_{\text{PIC}}^{(N)} = \mathfrak{G}_y \cap \mathfrak{G}_x = \mathfrak{G}_y$.

There is an exceptional case under which $\theta_2(z|\tau) = \pm\theta_1(z|\tau)$. The known property $\theta_1(z + \frac{1}{2}|\tau) = \theta_2(z|\tau)$ implies that this is the case $\theta_2(\pm\frac{1}{4}) = \pm\theta_1(\pm\frac{1}{4})$ and therefore $(\nu, \mu, N) = (0, \pm 1, \pm 2)$ (see Example 2 further below). There are two sets of parameters (ν, μ, N) in (40):

$$\theta\left(\frac{\nu\tau + \mu}{N}\right), \quad N = 2, 3, 4, \dots \quad (43)$$

and

$$\theta\left(2^k \frac{\nu\tau + \mu}{N}\right), \quad N = 3, 5, 7, \dots, \quad k = 1, 2, 3, \dots, \quad (44)$$

where ν and μ are not even simultaneously. Both of the cases (43), (44) come from Weierstrass' $\wp\left(\frac{\nu\tau + \mu}{N}|\tau\right)$ or $\wp\left(2^k \frac{\nu\tau + \mu}{N}|\tau\right)$ with the same meaning of ν, μ . Let R_N denote a rational function of \wp determining the multiplication theorem for $\wp(Nz)$. Therefore

$$R_N\left(\wp\left(\frac{\nu\tau + \mu}{N}|\tau\right)\right) = \wp(\nu\tau + \mu|\tau) = \dots \quad (45)$$

and this expression may have only one of the three values

$$\dots = \{\wp(1|\tau), \wp(\tau|\tau), \wp(\tau + 1|\tau)\}, \quad (46)$$

so that all the values ν, μ are equivalent to the three cases $\nu\tau + \mu = \{1, \tau, \tau + 1\}$:

$$\wp(1|\tau) = \frac{\pi^2}{12} \vartheta_3^4(\tau)(x+1), \quad \wp(\tau|\tau) = \frac{\pi^2}{12} \vartheta_3^4(\tau)(x-2), \quad \wp(\tau+1|\tau) = \frac{\pi^2}{12} \vartheta_3^4(\tau)(1-2x)$$

(hence, in order to get all PH-curves one needs to use only multiplication theorems⁶ for \wp). The two latter cases produce automorphisms \mathfrak{G}_y which are conjugated to the case $\wp(1|\tau)$ since $\wp(1|\tau) \mapsto \wp(\tau|\tau)$ under $\tau \mapsto -\frac{1}{\tau}$ and $\wp(1|\tau) \mapsto \wp(\tau + 1|\tau)$ under $\tau \mapsto \frac{\tau}{\tau+1}$.

⁶No need to involve addition formulae as mentioned in [52, Lemma 3]; multiplication formulae have nice and effective recurrences [60, III: p. 105] and are much more effective in use than additional ones.

Furthermore, in this case $\wp\left(2^k \frac{\nu\tau+\mu}{N} | \tau\right) = R_{2^k}\left(\wp\left(\frac{\nu\tau+\mu}{N} | \tau\right)\right)$ and we get a rational function of the algebraic one corresponding to the case $\wp\left(\frac{\nu\tau+\mu}{N} | \tau\right)$ with ν, μ equal to 0 or 1. Therefore genera of groups $\mathfrak{G}_{\text{Pic}}^{(N)}$ and $\mathfrak{G}_{\text{Pic}}^{(N)}$ coincide. Again, all the groups will be conjugated to the group with $(\nu, \mu) = (0, 1)$ and topological genus of polygon for $\mathfrak{G}_{\text{Pic}}^{(N)}$ coincides with the genus of the curve $F_{(N)}(x, y) = 0$.

Transformations (38) imply the following congruence conditions on entries of (39):

$$2^k(\nu n + \mu p) = NP, \quad 2^k(\nu m + \mu q) = NQ, \quad P, Q \in \mathbb{Z}$$

and, by previous argument, we may consider only the case $\nu\tau + \mu = 1$, where N is a free parameter being an odd number or $N \in \mathbb{Z}$ if $k = 0$. Therefore $P, Q \sim 2^k$ and we have $p, q \sim N$. The condition $\det \mathfrak{G} = 1$ yields $2N(mp - nq) = Nq + n$ and therefore $n \equiv 0 \pmod{N}$. Replacing n with Nn , we arrive at (42) and the first set \mathfrak{G} of matrices in (41). Above-mentioned transformations of \wp 's have been formed by the two ones: $\tau \mapsto \tau + 1$ and $\tau \mapsto -\frac{1}{\tau}$. Conjugating \mathfrak{G} by the first of them, we get \mathfrak{G} itself. The second one produces $(\mathfrak{G}^\tau)^{-1} = \mathfrak{G}^\tau$, that is the second set in (41).

There is a general algorithm concerning subgroups of free groups and it is known as the famous Reidemeister–Schreier rewriting process [48]. As applied to our automorphisms, the algorithm is somewhat too general since it deals with abstract group presentations but we are concerned with the matrix monodromies. Since $\mathfrak{G}_{\text{Pic}}^{(N)}$ is a subgroup of the free group $\Gamma(2)$, the monodromy \mathfrak{G}_y (in order to be free) has to have a parabolic element T_0 and this generator, being a generator of the *global* monodromy \mathfrak{G}_y , must correspond to punctures at $\mathbb{H}^+/\mathfrak{G}_y$ and $\mathbb{H}^+/\mathfrak{G}_x$. Obviously, T_0 is a power of some parabolic element from $\Gamma(2)$. We thus get a polygon for \mathfrak{G}_y as a set of copies of x -quadrangles. Generators of the global monodromy \mathfrak{G}_y are easily obtainable by geometric analysis of this ‘big’ polygon supplemented with solution of congruence (42). ■

The following table illustrates genera and degrees of the covers $x \mapsto y$ for some of the PH-curves; we took the maximal y -degree connected component of solutions determined by formulae (45)–(46):

| N | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | ... |
|-----------------------|---|---|---|----|----|----|----|----|----|----|----|----|-----|
| genus | 0 | 0 | 0 | 1 | 1 | 4 | 5 | 7 | 9 | 16 | 13 | 25 | ... |
| number of y -sheets | 2 | 4 | 8 | 12 | 16 | 24 | 32 | 36 | 48 | 60 | 64 | 84 | ... |

(47)

For prime N 's these genera fit in the Barth–Michel formula $g = \frac{1}{4}(N - 3)^2$ [5] and Hitchin did show that the formula works for algebraic solutions defined by Poncelet's polygons [37].

It remains to consider the exceptional case in Theorem 7. We shall do this more fully since this case demonstrates the way of getting formulae.

Example 2. Let us consider function $u(\tau)$ in (37). Imposing the invariancy condition for this ratio, we obtain equations on entries of the transformation:

$$i^{2p} = 1 \Rightarrow p = \{0, 2\}, \quad ad - bc = 1 \Rightarrow 2(mp - qn) = q + n. \quad (48)$$

Integral solutions to these equations completely determine the group $\mathfrak{G}_u = \mathbf{Aut} u(\tau)$, that is the monodromy group of Heun's equation for Legendre's modulus $u = k(\tau)$:

$$\Psi'' = -\frac{1}{4} \frac{(u^2 + 1)^2}{u^2(u^2 - 1)^2} \Psi. \quad (49)$$

It is obvious that we have to have three generators for this group and they are obtainable from (48). The group \mathfrak{G}_u is a freely generated one with index 2 in $\Gamma(2)$ and we need only find any three solutions of (48) being integers n, m, q nearest to zero. The pairing of neighboring quadrangles for $\Gamma(2)$ implies that we should choose only parabolic representatives. One easily finds:

$$\begin{aligned} \mathfrak{G}_u &= \langle T_{(1)}, T_{(0)}, T_{(\infty)} \rangle = \langle U, S^2, (SU^{-1})^2 \rangle \\ &= \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix} \right\rangle, \\ u(i\infty) &= 1, \quad u(0) = 0, \quad u(1) = \infty, \quad u\left(\frac{1}{2}\right) = -1, \end{aligned}$$

where we have represented T 's in form of decompositions by the standard $\Gamma(2)$ -generators: $U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. This matrix representation for monodromy of (49) has been attached to a certain basis of solutions (Ψ_1, Ψ_2) . Such a basis is calculated, as usual in uniformization, through the uniformizing Hauptmodul [27]:

$$\sqrt{u_\tau} = \sqrt{\frac{\pi}{2i} \frac{\vartheta_4^2}{\vartheta_3^2} \vartheta_2^4} \sim \Psi_1 = \sqrt{u(u^2 - 1)} K'(u), \quad \Psi_2 = i\sqrt{u(u^2 - 1)} K(u). \quad (50)$$

7. TOWERS OF UNIFORMIZABLE CURVES. EXAMPLES

We now pass to consequences. Having two curves at hand, $F_1(x, y) = 0$ and $F_2(x, \tilde{y}) = 0$, we may formally eliminate the variable x and get one more curve $F(y, \tilde{y}) = 0$. In particular, *any algebraic PH-solution is algebraically related to any other PH-algebraic solution*. Passing now from algebraic curves to their uniformizations $y(\tau), \tilde{y}(\tau)$, we see that such an elimination is compatible with parametrizations if and only if the one common function $x(\tau)$ appears in parametrizations of both of the curves, i.e. $x(\tau)$ is the universal uniformizing function for branch x -points of F_1 and F_2 . This seemingly trivial procedure, once transformed into the ‘ τ -representation’, leads to rather nontrivial consequences. Namely, new and completely uniformizable curves of higher genera. Indeed, let $x = x(\tau)$ be the universal uniformization for curves F_1 and F_2 and functions $y(\tau)$ and $\tilde{y}(\tau)$ are known. Then we obtain not only parametrization of the curve $F(y, \tilde{y}) = 0$, but Fuchsian equations, groups, and differentials as well. The general explanation of this fact is that automorphism group \mathfrak{G}_z of some modular function $z(\tau)$ has a lot of subgroups of higher genera even though modular equation defining the function $z(\tau)$ has a zero genus. In this respect the equation itself is just a particular case of this infinite family. We thus construct curves of nontrivial genera without seeking for complicated subgroups of $\Gamma(1)$ having nontrivial genera. For example, Klein's curves $F(J(\tau), J(N\tau)) = 0$ and their groups $\Gamma_0(N)$ have zero genera for N up to 10 and $\Gamma_0(25)$ has again the genus zero. See [50] for details and explicit formulae.

As numerous examples show, the curves obtained from PH-series are simpler than the majority of classical modular equations. In addition to all this, Theorem 5 provides a complete differential apparatus for these curves.

Example 3. The relation between $u(x) = \sqrt{x}$ and Picard's curve $y(x)$ of level $N = 5$ is the curve of genus $g = 5$:

$$\begin{aligned} & (16y^2 - 20y + 5)u^{12} - 2y(40y^2 - 47y + 10)u^{10} \\ & + y^2(64y^5 - 240y^4 + 360y^3 - 105y^2 - 80y + 16)u^8 \\ & - 20y^5(8y^3 - 28y^2 + 39y - 18)u^6 + 5y^6(28y^3 - 89y^2 + 112y - 48)u^4 \\ & - 2y^7(5y - 4)(5y^2 - 10y + 8)u^2 + y^{12} = 0. \end{aligned} \quad (51)$$

Uniformizing functions $u(\tau)$ and $y(\tau)$ for this curve read as follows

$$u = \frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)}, \quad y = -\frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)} \frac{\theta_2^2\left(\frac{1}{10}|\tau\right)}{\theta_1^2\left(\frac{1}{10}|\tau\right)}$$

and have monodromies of genus zero and unity respectively (see table (47)). The curve (51) is not hyperelliptic but can be realized as a cover of a torus. Indeed, equation (51) has the obvious sheet interchange symmetry $u \mapsto -u$ and therefore we may consider (51) as a cover of the plane u^2 , that is x -plane again. A simple calculation shows that this cover is an arithmetic torus equivalent to the form $w^2 = z^3 - 12z - 11$ having Klein's J -invariant equal to $\frac{256}{135}$. Incidentally it should be remarked that the above-written PH-parametrization of the torus is highly elementary, whereas its \wp -Weierstrass' one is too tremendous to display here.

7.1. Non-3-branch covers. As well as being a very wide class of completely describable curves, PH-curves provide a large number of curves of not so special form as themselves. Curves of this family are also of interest because these, contrary to Painlevé curves, have greater than three branch points and, therefore, do not belong to the well-known class of the Belyi curves. Formula (51) is a good example but even simpler PH-curves yield a rich theory with nice consequences.

Let us consider two simplest Picard's cases corresponding to $N = \{2, 3\}$ (by virtue of Corollary 1 we could equally well take Hitchin's formulae). They produce two solutions $u(x)$ and $y(x)$ defined by the curves (Hitchin–Dubrovin)

$$u^2 = x, \quad y^4 - 6xy^2 + 4x(x+1)y - 3x^2 = 0; \quad (52)$$

second of these solutions corresponds to $A\tau + B = \frac{1}{6}$. Solutions (52) lie on the elliptic curve

$$y^4 - u^2(6y - 4)y + u^4(4y - 3) = 0 \quad (53)$$

although both $u(\tau)$ and $y(\tau)$ have zero genus monodromies (see table (47) again). In order to show how general Fuchsian equations (27) may look we exemplify an equation for the function $y(\tau)$. From Theorem 3 we obtain:

$$[y, \tau] = -\frac{1}{8} \frac{8x(y-2)(y^2-9y+9)y+16y^6+27y^5+95y^4-415y^3+465y^2-288y+108}{y^2(4y-3)(y+3)^2(y-1)^3}$$

and this equation can serve as a nontrivial example of solvable Fuchsian equation (5) with algebraic coefficients. Turning back to the torus (53), we can transform it into any of standard forms. Manipulations of this kind $(y, u) \rightleftarrows (z, w)$ have long been algorithmized and we adopt Riemann's form

$$w^2 = z(z-1)(z+3) \quad (54)$$

obtainable from (53) by the following birational isomorphism

$$\begin{aligned} y &= -\frac{1}{4} \frac{w^2}{z}, & z &= 2 - 3 \frac{u^2}{y} - \frac{u^2 - 1}{y - 1}, \\ u &= \frac{1}{4} (1 - z^{-1}) w, & w &= \frac{(5y - 3)u^2 - y^3 - y^2}{u(u^2 - 1)}. \end{aligned} \quad (55)$$

That we have a torus in a canonical form does not mean that we arrive at a Fuchsian equation with singularities located precisely at four branch-places of canonical structures for tori. However the simplicity of (54) tells us that function z is certain to satisfy a simple Fuchsian equation. This is so indeed and we obtain (Lemma 1) that expression

$$z(\tau) = 2 + 3 \frac{\vartheta_4^2}{\vartheta_3^2} \cdot \frac{\theta_1^2}{\theta_2^2} - \frac{\vartheta_2^2}{\vartheta_3^2} \cdot \frac{\theta_1^2}{\theta_4^2}, \quad (56)$$

where $\theta := \theta\left(\frac{1}{6}|\tau\right)$, solves an equation which turns out to be very elegant:

$$[z, \tau] = -\frac{1}{2} \frac{(z^2 + 3)^4}{(z^5 - 10z^3 + 9z)^2} =: \mathcal{Q}(z) \quad (57)$$

(exercise: check this equation employing Theorem 5 and Lemma 2). This equation has regular singular points $E_k = \{0, \pm 1, \pm 3\}$ and $E_6 = \infty$ so that $z^5 - 10z^3 + 9z$ has $z = E_k$ for roots. Since

$$\mathcal{Q}(z) = -\frac{1}{2} \sum_{k=1}^5 \frac{1}{(z - E_k)^2} + \frac{2z^2 - 6}{(z - 1)(z + 1)(z - 3)(z + 3)},$$

all the six points $\{E_k\}$ correspond to punctures. Correlating this equation with torus (54), we could treat the set $\{E_k\}$ as the fact that torus (54) has 'superfluous' punctures at points $z = \{-1, 3\}$ but more correct formulation, as a continuation and illustration to Remark 1, reads as follows.

Remark 5. Monodromies and punctures are defined not by curves but Fuchsian equations. Therefore punctures, as attributes of functions and Poincaré τ -domains of their automorphisms, are not bound to be branch places of a certain curve. They may be located at any places on any curves, including even non-branch (regular) places. This is just the case of torus (54), whereas the function $z(\tau)$ itself and its equation (57) may uniformize many other curves with branch places $z = E_k$, say, the curves of the form⁷

$$v^m = z^n (z - 1)^p (z + 1)^q (z - 3)^r (z + 3)^s. \quad (58)$$

⁷The numbers $\{m, n, p, q, r, s\}$ are allowed to be any complex numbers so that we construct the formal parametrizations of *non-algebraic* dependencies (genus is not finite) by monodromies of finite topological

Thus, Painlevé *PH-curves* generate new universal uniformizations forming towers of new curves. The function (56), for example, uniformizes all the curves of the form (58) in the sense that analytic function $v(\tau)$ determined by the relations (56), (58) is a single-valued function in the entire domain of its existence, that is \mathbb{H}^+ . We can even enlarge this class by the change $z \mapsto \sqrt{z}$. This is possible because of Picard's function $y(\tau)$ is a perfect square and, from the first formula in (55), one follows that z is also the perfect square:

$$z = -y \frac{(z-1)^2}{4u^2} \Rightarrow z(\tau) = \left\{ (z(\tau)-1) \frac{\vartheta_3 \theta_2}{2\vartheta_4 \theta_1} \right\}^2$$

(non-obvious fact for θ -constant expression (56)). Therefore Hauptmodul $r = \sqrt{z(\tau)}$ satisfies the equation (Lemma 1)

$$[r, \tau] = 4r^2 \mathcal{Q}(r^2) + \frac{3}{2} \frac{1}{r^2}$$

(exercise: describe its singularities). We shall return to the example (58) further below since one of its particular cases is related to the widely known Jacobi's curves

$$v^2 = z(z-1)(z-a)(z-b)(z-ab). \quad (59)$$

Appearance of such curves is not an exception and we observe in passing that the way of getting the hyperelliptic formulae from Picard–Hitchin's towers is simpler than the direct search for hyperelliptic modular equations, say of genus $g = 2$, among subgroups of known congruence groups of level N . In all the known cases the level turns out to be very large; see for example analysis of group $\Gamma_0(50)$ in work [10]. We consider yet another example because it is connected with very classical objects.

7.2. Schwarz hyperelliptic curve. The structure of Picard's solutions is such that we may apply to any of them the substitution $\{x = p^4, y = -p^2q\}$. Consider again the small values of N and carry out this substitution in the second of the curves (52) ($N = 3$). We then obtain the genus $g = 3$ curve

$$4(p^2 + p^{-2})q = q^4 - 6q^2 - 3, \quad \left\{ p = \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}, \quad q = \frac{\theta_2^2(\frac{1}{6}|\tau)}{\theta_1^2(\frac{1}{6}|\tau)} \right\}. \quad (60)$$

The structure of (60) tells us that it should be hyperelliptic and a simple calculation shows that it is isomorphic to the classical Schwarz curve

$$y^2 = x^8 + 14x^4 + 1. \quad (61)$$

This curve repeatedly appears throughout both volumes of Schwarz's *Gesammelte Abhandlungen* [59] in different contexts⁸ but none of parametrizations was known hitherto.

genus! In this regard even τ -forms for non-algebraic (not finitely sheeted covers of x -plane) but single-valued solutions $y(\tau)$ to equation \mathcal{P}_6 itself provide a large number of such examples. Apart from PH-solutions (20), (23) here is a simplest one (new?): for arbitrary $s \in \mathbb{C}$ the function $y = x^s$ solves (1) when $(\alpha, \beta, \gamma, \delta) = (0, 0, s^2, (s-1)^2 - \frac{1}{2})$. See also comments as to work by Guzzetti [34] in Sect. 6 of work [15]. Do check that the simple generalization $y = x^s(x-1)^r$ does not exist for $s, r \notin \mathbb{Q}$.

⁸Including the very 1867 work [59, I: p. 13] wherein his famous Schwarz derivative $\Psi(s, u)$ has arisen.

Birational transformation between (60) and (61) is as follows

$$\begin{aligned} p &= \frac{\mathbf{x}^4 - \mathbf{y} - 1}{4\mathbf{x}}, & \mathbf{x} &= \frac{p(q^2 - 1)}{2(p^2 + q)}, \\ q &= \frac{\mathbf{x}^4 - \mathbf{y} + 1}{2\mathbf{x}^2}, & \mathbf{y} &= \frac{q^2 + 3}{q^2(q^2 - 1)}(4p^2q + q^2 + 3). \end{aligned} \quad (62)$$

One easily obtains parametrizations and Fuchsian equation for $\mathbf{x}(\tau)$. Curiously, in doing so we arrive at further enlargement of the tower for curve (61). This is a frequently encountered situation in the universal uniformization. Since the objects are very classical, we state the results as a separate proposition.

Proposition 3. *The function $\mathbf{x} = \mathbf{x}(\tau)$ defined by ϑ, θ -ratio (60), (62) has a zero/pole divisor determined by the quotient*

$$\mathbf{x} = \frac{\hat{\eta}^3(\tau)\theta_2\left(\frac{1}{3}|\tau\right)}{\theta_1^2\left(\frac{1}{6}|\tau\right)\theta_3^2\left(\frac{1}{6}|\tau\right)} \quad (63)$$

and uniformizes the tower of curves formed by Burnside's tower $\mathbf{z}^n = \mathbf{x}^5 - \mathbf{x}$ and Schwarz's curve (61):

$$\mathbf{z}^2 = (\mathbf{x}^8 + 14\mathbf{x}^4 + 1)(\mathbf{x}^5 - \mathbf{x}). \quad (64)$$

Proof. The sequential algebraic changes of variables $x \mapsto p \mapsto \mathbf{x}$ defined by (62) transform (Lemma 1) equation (9) into the following Fuchsian equation:

$$\begin{aligned} [\mathbf{x}, \tau] &= -\frac{1}{2} \frac{\mathbf{x}^{24} - 102\mathbf{x}^{20} + 1167\mathbf{x}^{16} + 1964\mathbf{x}^{12} + 1167\mathbf{x}^8 - 102\mathbf{x}^4 + 1}{(\mathbf{x}^8 + 14\mathbf{x}^4 + 1)^2(\mathbf{x}^5 - \mathbf{x})^2} \\ &= -\frac{3}{8} \sum_{\beta} \frac{1}{(\mathbf{x} - \beta)^2} - \frac{1}{2} \sum_{\alpha} \frac{1}{(\mathbf{x} - \alpha)^2} + \frac{(\mathbf{x}^4 + 7)(5\mathbf{x}^4 - 1)\mathbf{x}^3}{(\mathbf{x}^8 + 14\mathbf{x}^4 + 1)(\mathbf{x}^5 - \mathbf{x})}, \end{aligned} \quad (65)$$

where summations run over roots of polynomials $\beta^8 + 14\beta^4 + 1$ and $\alpha^5 - \alpha$. This proves uniformization of curves (61), (64) and defines behavior of solutions at all the branch places.

In order to present $\mathbf{x}(\tau)$ defined by (62) in form of the ratio of prime forms (63) one needs to use the duplication formula $\theta_2^4(z) - \theta_1^4(z) = \vartheta_2^3 \cdot \theta_2(2z)$, standard quadratic θ -identities, and second relation in (13). (Exercise: derive independently of birational transformations (62), with use of Theorem 5, that function (63) satisfies an equation with rational coefficients and this equation is (65)). \blacksquare

The polynomial expressions above, written in homogeneous form $z_1^8 + 14z_1^4z_2^4 + z_2^8$, $z_1z_2(z_1^4 - z_2^4)$, are widely known as Schwarz–Klein's ground forms representing symmetry group of octahedron [59, 28, 38]. We thus obtain that correlation of two simplest PH-curves (52) gives a uniformizing τ -representation for their production.

Example 4 (exercise). Find 13-generated monodromy representation $\mathfrak{G}_{\mathbf{x}} = \langle T_k \rangle$ for the Schwarz–Burnside equation (65) with use of ϑ, θ -representation (63) and Theorem 6. Compute the base of corresponding solutions $\Psi_{1,2}(\mathbf{x})$ attached to Hauptmodul (63).

7.3. Some conclusions. Summarizing the preceding material, we may formulate the following general recipe:

- Every algebraic curve (3) may be thought of as an algebraic substitution/transformation $x \mapsto y$ in any Fuchsian equation $[x, \tau] = \mathcal{Q}(x, y)$. Conversely, any rational/algebraic substitution $x \mapsto z : x = \Phi(z)$ (or $\Phi(x, z) = 0$) may be thought of as an algebraic curve generating the new curve $\Xi(y, z) = 0$. Let the monodromy \mathfrak{G}_x have a finite genus, i.e. correct accessory parameters in the equation $[x, \tau] = \mathcal{Q}(x, y)$, and uniformize the curve (3). Suppose the *local* ramifications $z = z(x)$ are compatible with the local behavior $x = E + a\tau^n + \dots$ in the sense that $z = E' + b\tau^m + \dots$, where $m \in \mathbb{Z}$. Then the *global* monodromy \mathfrak{G}_z (and \mathfrak{G}_y of course) has also finite genus, i.e. correct accessory parameters in the proper Fuchsian equations $[z, \tau] = \tilde{\mathcal{Q}}(z, y; x)$. The universal uniformization is characterized by that it ‘covers’ the arbitrary ramification orders; its tower is thus always infinite.

At this point, it is worth noting that the ‘trivial’ (zero genus) rational substitutions like $x = R(z)$ may lead to nontrivial curves of increasing genera and, on the other hand, nontrivial substitutions $\Phi(x, z) = 0$ of higher genera may preserve the zero genus of monodromies for both \mathfrak{G}_x and \mathfrak{G}_z , or \mathfrak{G}_y . Insomuch as all these transitions constitute just substitutions, all the appearing Fuchsian equations are integrable one through another. In particular, we have the following conclusion.

Theorem 8. *Fuchsian equations (27) corresponding to any algebraic Painlevé uniformizing functions $y(\tau)$ are pullbacks of hypergeometric ${}_2F_1$ -equations by rational or algebraic functions. These ${}_2F_1$ -integrabilities form an equivalence relation; for example, equivalence with equation defining the series ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1|z)$.*

The next section contains further nontrivial examples along these lines.

8. INVERSION PROBLEMS AND RELATED TOPICS

Uniformization of PH-curves leads to many remarkable consequences and there is no escape from the mentioning some very well-known Fuchsian equations.

8.1. Exceptional cases of Heun’s equations. The symmetry $z \mapsto -z$ of equation (57) suggests to make the substitution $z^2 = s$ and to expect some simplification of this equation. Doing this, we obtain that function $s(\tau)$ satisfies the ODE

$$[s, \tau] = -\frac{1}{2} \frac{s^4 - 12s^3 + 102s^2 - 108s + 81}{s^2(s-1)^2(s-9)^2}$$

which, in the language of the ‘linear Fuchs theory’, corresponds to

$$\Psi'' = -\frac{1}{4} \left(\frac{1}{s^2} + \frac{1}{(s-1)^2} + \frac{1}{(s-9)^2} - \frac{2s-8}{s(s-1)(s-9)} \right) \Psi. \quad (66)$$

It is a Heun equation with singularities at $s = \{0, 1, 9, \infty\}$ defining the monodromy of a 4-punctured sphere since exponent differences at all the singularities are equal to zero. It is notable that this equation is not treated by any classical algorithmic methods in theories of integration of linear ODEs (the Picard–Vessiot theory [58]). This fact is not strange. Presently only four exceptional cases of integrable Fuchsian equations with such

a type of parabolic singularities are known. They were claimed by D. Chudnovsky & G. Chudnovsky in work [17] and read in their notation as follows [17, p. 185; correcting a typo for case (III)]

$$x(x^2 - 1)y'' + (3x^2 - 1)y' + xy = 0, \quad (\text{I})$$

$$x(x^2 + 3x + 3)y'' + (3x^2 + 6x + 3)y' + (x + 1)y = 0, \quad (\text{II})$$

$$x(x - 1)(x + 8)y'' + (3x^2 + 14x - 8)y' + (x + 2)y = 0, \quad (\text{III})$$

$$x(x^2 + 11x - 1)y'' + (3x^2 + 22x - 1)y' + (x + 3)y = 0. \quad (\text{IV})$$

All these cases are pullbacks of hypergeometric functions by rational maps though no their Hauptmoduln and monodromies have been tabulated in the literature. The case (I) is equivalent to equation (49); we have detailed it in [Example 2](#). The case (III) becomes equation (66) after the change

$$x = 1 - \frac{9}{s}. \quad (67)$$

There is yet another point (not only) showing an exceptional feature of these equations. This is a complete set of stable families of elliptic curves over \mathbb{P}^1 with four singular fibres found in the nice note [7] (sometimes referred to as Beauville's curves; see also [50] for relevant information). Translation of this into our language means that there are only six zero genus orbifolds $\mathbb{H}^+/\mathfrak{G}_x$, where $\mathfrak{G}_x \subset \Gamma(1)$, with exactly four cusps and no elliptic points. The simplest relation to the four ODEs written above is as follows.

Take for example the elliptic curve

$$(X + Y)(Y + Z)(Z + X) + tXYZ = 0 \quad (68)$$

which is the fourth number in Beauville's table (notation as in [7, p. 658]; in the same place see information about groups). Its J -invariant is obviously a rational function of the parameter t . Considering this dependence $J = R(t)$ as a change of variable in Fuchsian $\Gamma(1)$ -equation

$$[J, \tau] = -\frac{1}{72} \frac{36J^2 - 41J + 32}{J^2(J - 1)^2}, \quad (69)$$

we arrive (Lemma 1) exactly at a normal form to equation (III) by replacing $t \mapsto x$. It thus becomes a Picard–Fuchs equation for Beauville's family (68) parametrized by a certain congruence subgroup \mathfrak{G}_t . The similar treatment takes place for other Beauville's curves and the same scheme as above integrates the Chudnovsky equations (I)–(IV) (see Remark 6 further below).

Explicit formulas for integrability of equation (66) in terms of ${}_2F_1$ follow immediately from [Sect. 7.1](#). Since s is algebraically related to $u = \sqrt{x} = k'(\tau) = k\left(\frac{-1}{\tau}\right)$, we obtain from (52) and (55):

$$64(2x - 1)^2s = (s^2 - 6s - 3)^2. \quad (70)$$

Therefore this algebraic change $s \mapsto x$, supplemented with the linear change $\Psi \mapsto \psi$:

$$\Psi = \sqrt{s_x} \cdot \psi(x) \implies \Psi = \frac{\sqrt[4]{s^3}}{s - 1} \cdot \psi(x), \quad (71)$$

reduces (66) to a hypergeometric equation in normal form (9): $\psi_{xx} = -\frac{1}{4} \frac{x^2 - x + 1}{x^2(x-1)^2} \psi$. We get finally

$$\Psi = \sqrt[4]{s(s-1)^2(s-9)^2} \cdot \{AK(\sqrt{x}) + BK'(\sqrt{x})\}, \quad (72)$$

where

$$x = \frac{1}{16} (3\sqrt[3]{s} + 1) (\sqrt{s} - 1)^3.$$

Recall that symbols K and K' represent hypergeometric functions (19) and all their analytic continuations.

It is clear that further examples are rapidly multiplied and, what is more important, *for all equations we have always had solutions to the inversion problems since these solutions are hidden forms of the one basic relation (18)*. E.g., for Heun's equation (66) inversion of the ratio of its two solutions

$$\frac{\Psi_1}{\Psi_2} = \frac{K\left(\frac{1}{4}\sqrt{(s-6)\sqrt{s}} - 3\sqrt[3]{s} + 8\right)}{K'\left(\frac{1}{4}\sqrt{(s-6)\sqrt{s}} - 3\sqrt[3]{s} + 8\right)} = \frac{a\tau + b}{c\tau + d}, \quad (73)$$

that is function $s = s\left(\frac{a\tau+b}{c\tau+d}\right)$, is a square of the θ -constant expression (56) with an appropriate choice of constants (a, b, c, d) . In order to determine them it is sufficient to consider any three points τ corresponding to any tori with complex multiplication since such tori have exact (algebraic over \mathbb{Q} [62]) values of K . The function $s(\tau)$ thus constitutes one further universal uniformizing function for a new set of points. Any algebraic functions of $s(\tau)$ with arbitrary ramifications at points $s = \{0, 1, 9, \infty\}$ is a single-valued functions of τ . The complete solution to inversion problem (73) and τ -representation for the Ψ -function will be given in Sect. 8.3.

Expression (72) is not the only form to solution⁹; every form depends on which group (variable) is chosen. For instance, involving the group $\Gamma(1)$, and therefore Klein' invariant J , we know that

$$J = \frac{4}{27} \frac{(x^2 - x + 1)^3}{x^2(x-1)^2}.$$

Correlating this expression with formula (70), we get one more integrating change and nonstandard representation for Klein's $J(\tau)$ through the ϑ, θ -constants (56).

Proposition 4. *Klein's invariant $J(\tau)$ has the following θ -constant representation*

$$J = \frac{1}{12^3} \frac{(s+3)^3 ((s-5)^3 + 2^7)^3}{s(s-1)^6(s-9)^2}, \quad (74)$$

where $s = s(\tau)$ is a square of expression (56) (see also formula (82) below).

Substitution (74) solves (66) in terms of hypergeometric functions ${}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{2}{3}; J\right)$ ¹⁰ and these functions are directly related to the classical representations for $J(\tau)$ through

⁹Another example is $\Psi(s) = \sqrt[6]{s-1} \sqrt{s(s-9)} {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1 \mid -\frac{s(s-9)^2}{27(s-1)^2}\right)$. The author is indebted to M. van Hoeij for this elegant solution.

¹⁰This is also true for arbitrary Painlevé curves at all. According to Theorem 8, any Fuchsian equation arising from Painlevé substitution (15) is integrable in terms of this ${}_2F_1(J)$ -series or $K, K'(\sqrt{x})$.

$\vartheta(\tau)$'s or Dedekind's function $\widehat{\eta}(\tau)$ and $\widehat{\eta}(\tau)$ itself is just a hidden form of the known θ -constant

$$\widehat{\eta}(\tau) = -ie^{\frac{\pi i\tau}{3}}\theta_1(\tau|3\tau).$$

There are of course many other analogs of (74) and therefore PH-curves and their consequences provide a further and rich development of theories to the classical ϑ - and $\widehat{\eta}$ -constants. As for solutions to the Ψ , Picard–Hitchin's hierarchy leads to a massive generalization of numerous quadratic and cubic transformations of ${}_2F_1$ -series listed in dissertation by Goursat [32]. In addition to this, we have also representations of groups, θ -Hauptmoduln, their differential calculus, etc.

The aforesaid and examples are the illustrations to what we said in the end of Sect. 2.3: substitutions bind integrable equations between themselves.

Corollary 3. *All the four Chudnovsky's equations are equivalent to each other in the sense that they are transformable into one another by algebraic changes of independent variable.*

The normal form (5), say, to the first equation of this list, is exactly equation (49) followed by renaming $u \mapsto x$. Remember now the relation (67) between variable x in equation (III) and our variable s . Changing $x \mapsto u^2$ in (70) followed by renaming $u \mapsto x$, we obtain that Chudnovsky's equations

$$x(x^2 - 1)y'' + (3x^2 - 1)y' + xy = 0, \quad (75)$$

$$x(x-1)(x+8)y'' + (3x^2 + 14x - 8)y' + (x+2)y = 0$$

are transformed into each other under the change

$$(2x^2 - 1)^2 = \frac{(x^2 - 20x - 8)^2}{64(1-x)^3} \quad (76)$$

which in turn is found to be a genus 1 curve. This formula can also serve as an illustration to Theorem 8. Transformations between the y 's in Fuchsian equations are always linear and easy computation shows that

$$y = (x-1)^{\frac{4}{3}}\bar{y}.$$

Remark 6 (exercises). Do transform other equations of the list into each other and compute genera of these substitutions. Hint: use the six Beauville's curves, their J -invariants, and Lemma 1. Show that these six curves [7, p. 658] lead to six Fuchsian equations for \mathfrak{G}_t (t is Beauville's notation as in (68)), among which are trivially equivalent. Namely, 1st and 6th curves of Beauville's list yield coinciding \mathfrak{G}_t -equations and \mathfrak{G}_t -equations for the 2nd and 5th curves differ by a trivial scale $t \mapsto 4it$. In the end this leads to the four independent Chudnovsky's equations. Their monodromies are not conjugate in $SL_2(\mathbb{R})$ but are commensurable in $\Gamma(1)$. This fact does not mean however that transformation between t -variables for, say, the 1st and 6th Beauville curves is trivial. They have the same Fuchsian equation and conjugated zero genus monodromies \mathfrak{G}_t 's but their t -variables lie on an equi-anharmonic elliptic curve. The 1st and 5th t -variables, on the other hand, are related by a complicated non-hyperelliptic curve of genus $g = 5$ (list these curves).

8.2. Apéry's differential equations. Roger Apéry, in his celebrated proof [3, 57] of the irrationality of Riemann's $\zeta(3) = 1^{-3} + 2^{-3} + 3^{-3} + \dots$, used the recursion

$$n^3 C_n = (34n^3 - 51n^2 + 27n - 5)C_{n-1} - (n-1)^3 C_{n-2}$$

and pointed out that it corresponds to the linear 3rd order ODE

$$r^2(r^2 - 34r + 1)\psi''' + r(6r^2 - 153r + 3)\psi'' + (7r^2 - 112r + 1)\psi' + (r - 5)\psi = 0 \quad (77)$$

with the help of standard correlation between recursions and linear ODEs: $\psi = \sum C_n r^n$. He also observed that this ODE was a second symmetric power of the following ODE of Fuchsian class (see also [24]):

$$r(r^2 - 34r + 1)\varphi'' + (2r^2 - 51r + 1)\varphi' + \frac{1}{4}(r - 10)\varphi = 0. \quad (78)$$

Uniqueness of Apéry's equation (77) suggests to seek for its relatives among some 'good' equations, i.e. equations with known monodromies, ${}_2F_1$ -integrals, etc. By virtue of pointed out relation between (77) and (78) we may restrict our consideration to equation (78) since solution of (77) is given by the formula $\psi = a\varphi_1^2 + b\varphi_1\varphi_2 + c\varphi_2^2$. Without entering into details of irrationality, integrality of Apéry's sequences, etc, we note that relation of these sequences to modular forms for certain subgroups of $\Gamma(1)$ was established in the beautiful paper of Beukers [9]. We shall give an independent motivation and explanations in the context of Picard–Hitchin's uniformization.

Let us transform (78) into the normal (and unique) Klein's form $\varphi'' = \frac{1}{2}\mathcal{Q}(r)\varphi$ by the linear change $\varphi \mapsto \sqrt[4]{r^4 - 34r^3 + r^2}\varphi$. We get Heun's equation

$$\varphi'' = -\left\{\frac{1}{2}\frac{1}{r^2} + \frac{3}{8}\sum_{\alpha}\frac{1}{(r-\alpha)^2} - \frac{3}{4}\frac{r-16}{(r^2-34r+1)r}\right\}\varphi, \quad (79)$$

where α 's are roots of equation $\alpha^2 - 34\alpha + 1 = 0$. Exponent differences δ for this equation are $\delta = 0$ at points $r = \{0, \infty\}$ and $\delta = \frac{1}{2}$ at $r = \alpha$'s.

Motivated by a desire to reduce (79) to some integrable form, we should try a transformation $r \mapsto s$ that must be of algebraic/rational type lest the Fuchsian class be escaped. The one-to-one linear fractional transformation will nothing yield and we try the rational one at first: $r = R(s)$. Obviously, images of two parabolic points $r = \{0, \infty\}$ will remain parabolic ones in s -equation under such a transformation since at these points we have

$$\frac{\varphi_1}{\varphi_2} \sim \ln r + \dots = \ln R(s) + \dots; \quad (80)$$

so at least two parabolic singularities are not removable. Considering other singularities, we conclude that if the transformation $r \mapsto s$ were regular at $r = \alpha$, it would cause the total number of s -singularities to increase which is undesirable. Indeed,

$$\frac{\varphi_1}{\varphi_2} \sim \sqrt{r-\alpha} + \dots = \sqrt{R(s)-\alpha} + \dots$$

and analytic function $R(s)$ has greater than two α -points. Therefore each point $r = \alpha$ must get mapped into one s -point, that is $s \sim \sqrt{(r-\alpha_1)(r-\alpha_2)}$, and points α themselves map into regular points of s -equation. New singularities can arise only from (80), all of them

will be again parabolic, and equation will of inevitably determine a punctured sphere¹¹. In other words, $R(s)$ should be a quadratic transformation of the form

$$r = \frac{(as-b)(s+c)}{(s-d)(s-e)}$$

having s -discriminant $\Delta = r^2 - 34r + 1$. Three s -images of singularities can be freely appointed so that, without loss of generality, we may send $(r = \infty) \mapsto (s = \{0, 1\})$ and $(r = 0) \mapsto (s = \infty)$. Other s -images are not ‘stirrable’ as we must satisfy the discriminant condition:

$$\left\{ r = \frac{as-b}{s(s-1)}, \quad \Delta = r^2 - 34r + 1 \right\} \Rightarrow \{a^2 = 1, 2b-a = 17\}.$$

We thus get

$$r = \frac{s-9}{s(s-1)} \quad \text{or} \quad r = \frac{s+8}{s(1-s)}. \quad (81)$$

It is a remarkable fact that in both of these cases we arrive at (66), so that Apéry’s equation turns out to be a hidden consequence of the two simplest but nontrivial solutions (52) coming from an infinite series of Picard’s ones. Indeed, the change $\varphi(r) = \sqrt{r_s} \Psi(s)$ and first of equalities (81) substituted in (79) cause this equation to become (66). The second equality produces an equivalent equation. This and other equivalents are obtained from (66) by the six linear fractional transformations $s \mapsto \frac{as+b}{cs+d}$ permuting points $s = \{0, 1, \infty\}$ between themselves and therefore the fourth singularity can be freely appointed to one of the six values $s = \left\{9, \frac{1}{9}, \frac{8}{9}, \frac{9}{8}, -8, -\frac{1}{8}\right\}$. Formulae (81) solve the inversion problem for Apéry’s equation (78).

8.3. Hauptmoduln and the Ψ (examples). Having a structure description of Heun–Apéry’s equations (66), (78)–(79) we can now involve simplifications with use of transformations of ϑ, θ -constants [62, 42] and sum up the preceding stuff. We shall do this without entering into details of calculations. For example, it becomes immediate to complete formula uniformization for equation (66) and related equations (57), (59).

Theorem 9. *The square of Hauptmodul (56)*

$$s(\tau) = 9 \frac{\vartheta_3^4(3\tau)}{\vartheta_3^4(\tau)} = \frac{\theta_1^4\left(\frac{1}{3}|\tau\right)\theta_3^4\left(\frac{1}{6}|\tau\right)}{\theta_2^4\left(\frac{1}{6}|\tau\right)\theta_4^4\left(\frac{1}{6}|\tau\right)} \quad (82)$$

solves the inversion problem (73) for Heun’s equation (66). The three group generators

$$\mathfrak{G}_s = \left\langle T_{(0)} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}_{\tau=\frac{1}{3}}, \quad T_{(\infty)} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}_{\tau=1}, \quad T_{(9)} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}_{\tau=\infty} \right\rangle \quad (83)$$

¹¹We have actually motivated here an important transition from non-free monodromy \mathfrak{G}_r to a free one \mathfrak{G}_s and obtained thereby yet another explanation as to why universal uniformization by free groups leads to large ‘towers of solvable curves’. As a rough guide one may think that *free* monodromy, being a rather large group, integrates many other equations but not only its proper one. The *larger* groups are *easier* described. For example group \mathfrak{G}_s is larger than \mathfrak{G}_r because it is completely free of defining relations, whereas \mathfrak{G}_r has the two ones $a^2 = b^2 = 1$.

determine the monodromy representation for equation (66) and automorphism of (82). Here, subscripts indicate the fix τ -points of T 's and values of s -singularities at them. The fourth point $s(0) = 1$ corresponds to the cycle of cusps

$$\tau = 0 \xrightarrow{T_{(0)}} \frac{1}{2} \xrightarrow{T_{(\infty)}} 2 \xrightarrow{T_{(9)}} 0,$$

so that $T_{(9)}T_{(\infty)}T_{(0)} = \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix} = T_{(1)}$.

Some consequences suggest themselves. From (70)–(71) we obtain (again, after ϑ, θ -simplifications) an explicit τ -representation of the function $\Psi(s)$.

Corollary 4. *The Heun equation (66) has the following uniformizing τ -representation for its solutions $\Psi_{1,2}(s)$ attached to Hauptmodul (82):*

$$\sqrt{\dot{s}} \sim \Psi_1(s) = \frac{\vartheta_3^3(3\tau)}{\vartheta_3(\tau)} \frac{\vartheta_2^4\left(\frac{\tau+1}{2}\right)}{9\vartheta_3^4(3\tau) - \vartheta_3^4(\tau)}, \quad \Psi_2(s) = \tau \Psi_1(s). \quad (84)$$

The relation (84) is an analog of the well-known $\Gamma(2)$ -Jacobi τ -representation for the elliptic integral $K(k) = \frac{\pi}{2} \vartheta_3^2(\tau)$. Notice incidentally that this identity and commensurability of all the groups in question in $\Gamma(1)$ imply that all such identities are merely hidden subgroup τ -representations of the general and well-known transformation law

$$\widehat{\eta}^2(\tau) \xrightarrow{\widehat{\Gamma(1)}(\tau)} \aleph^{\frac{2}{3}} \cdot (c\tau + d) \widehat{\eta}^2(\tau).$$

It is in turn a hidden form of solution to the known linear Fuchsian equation with $\Gamma(1)$ -monodromy

$$J(J-1)\psi'' + \frac{1}{6}(7J-4)\psi' + \frac{1}{144}\psi = 0$$

(an equivalent of equation (69)) and therefore is a corollary of the obvious fact

$$\psi \xrightarrow{\mathfrak{G}_J = \Gamma(1)} (c\tau + d)\psi.$$

It follows immediately that $\psi = \widehat{\eta}^2(\tau)$ (check this directly).

Remark 7 (exercise). It is well known that differential calculus associated with Ψ -functions for Legendre's modulus $k^2(\tau)$ contains the closed set of functions $\{K, K', E, E'\}$; the derivatives $\frac{dK}{dk}, \frac{dE}{dk}, \dots$ are functions of K, E 's themselves [26], [60, IV: p. 157]:

$$\begin{aligned} \frac{dK}{dk} &= -\frac{K}{k} - \frac{E}{(k^2 - 1)k}, & \frac{dK'}{dk} &= \frac{kK'}{1 - k^2} + \frac{E'}{(k^2 - 1)k}, \\ \frac{dE}{dk} &= -\frac{K}{k} + \frac{E}{k}, & \frac{dE'}{dk} &= \frac{kK'}{1 - k^2} + \frac{kE'}{k^2 - 1}. \end{aligned} \quad (85)$$

As a references source, we present here also the modular representations for $\{E, E'\}$ since no complete set of such formulae seems to have been tabulated in the standard texts (see also comments as to classical elliptic integrals in [49]). Formulae for K and K' are known

$$K(k) = \frac{\pi}{2} \vartheta_3^2(\tau), \quad K'(k) = \frac{\pi}{2i} \tau \vartheta_3^2(\tau), \quad k = \frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)} \quad (86)$$

and formulae for E, E' read

$$\begin{aligned} E(k) &= \frac{2}{\pi} \frac{1}{\vartheta_3^2(\tau)} \left\{ \eta(\tau) + \frac{\pi^2}{12} [\vartheta_3^4(\tau) + \vartheta_4^4(\tau)] \right\}, \\ E'(k) &= \frac{2}{\pi} \frac{i}{\vartheta_3^2(\tau)} \left\{ \tau \eta(\tau) - \frac{\pi^2}{12} [\vartheta_2^4(\tau) + \vartheta_3^4(\tau)] \tau - \frac{\pi i}{2} \right\}. \end{aligned} \quad (87)$$

Lemma 2 closes all the differential computations related to the classical objects $\{K, K', E, E'\}$ in both k - and τ -representations. In a reverse direction the formulae read [2, 18.9.13]

$$\tau = i \frac{K'(k)}{K(k)}, \quad \eta(\tau) = K(k)E(k) + \frac{1}{3}(k^2 - 2)K^2(k). \quad (88)$$

The exercise here is a derivation of the analog to these rules for Heun's group \mathfrak{G}_s and (84). In other words, these rules, as a complete set of the associated Picard–Fuchs equations, will be an ‘ s -version’ of the following closed chain written in the ‘ τ -representation’:

$$\dot{s} \rightsquigarrow \vartheta, \theta\text{-forms}, \quad \ddot{s} \rightsquigarrow \vartheta, \theta\text{- and } \eta\text{-form}, \quad \dddot{s} \rightsquigarrow \mathcal{Q}(s). \quad (89)$$

Section 10 is devoted to explanations of this chain in a more general context.

We close this section with a hyperelliptic example promised in Sect. 7.1. From equations (71) and (72) it follows that

$$\frac{16s}{s-1} \sqrt{x(x-1)} = \pm \sqrt{s(s-1)(s-9)}.$$

Remembering now that $s = z^2$, we see that this relation leads us to the hyperelliptic subcase of the family (58), that is (59).

Proposition 5. *The hyperelliptic curve*

$$\begin{aligned} v^2 &= z^5 - 10z^3 + 9z \\ &= z(z-1)(z+1)(z-3)(z+3) \end{aligned} \quad (90)$$

is described by Fuchsian equation (57) and has the following parametrization:

$$z = 3 \frac{\vartheta_3^2(3\tau)}{\vartheta_3^2(\tau)}, \quad v = 48\sqrt{3}i \frac{\vartheta_3^3(3\tau)}{\vartheta_3^3(\tau)} \frac{\vartheta_2^2(\tau)\vartheta_4^2(\tau)}{9\vartheta_3^4(3\tau) - \vartheta_3^4(\tau)}.$$

Example 5 (exercise). Find representation for genus $g = 2$ hyperelliptic uniformizing group \mathfrak{G} of the curve (90) and continue tower for it. Hint: for general recipe to parabolic uniformization of hyperelliptic Hauptmoduln see [14, Sect. 6.2].

Example 6 (exercise). For the curve (90) do construct the ϑ, θ -representations for $\sqrt{z \pm 1}$, $\sqrt{z \pm 3}$ as analogs of Weierstrassian single-valued functions $\sqrt{\wp - e_k} = \sigma_k$. Simplify (90) into the curve $v^2 = z^6 - 1$.

9. RELATIONS TO PAINLEVÉ NON-PH-CURVES

Our constructions considered up until this point were dealing with PH-curves being Painlevé curves or their consequences being no solutions to \mathcal{P}_6 . We can, however, get interesting information involving Painlevé non-PH-curves.

Consider a genus zero Painlevé non-PH-curve $F(x, y) = 0$. It has a rational parametrization $x = R_1(T)$, $y = R_2(T)$. On the other hand, by virtue of universality of $x(\tau)$, it has parametrization $y = y(\tau)$ through the ‘universal’ τ , wherein $y(\tau)$ is as yet unknown. Rational parametrization through the T tells us that T itself is a rational function of x, y , that is $T = R(x, y)$. Hence T becomes a univalent function of τ and therefore must satisfy a Fuchsian ODE with the rational $\mathcal{Q}(T)$ -function:

$$T = R(x(\tau), y(\tau)) \quad \Rightarrow \quad [T, \tau] = \mathcal{Q}(T).$$

In general, this way leads to *new* universal Hauptmoduln $T(\tau)$ with monodromies known to be Fuchsian. We may further take one more curve $\tilde{F}(x, \tilde{y}) = 0$ (PH-curve, say) with a known Hauptmodul $\tilde{y}(\tau)$. Rational uniformizer \tilde{T} for this second curve and the old one T can lie on a curve $\Xi(T, \tilde{T}) = 0$ of a quite high genus. But if \tilde{T} is a linear fractional function of T then y becomes a rational function on the second curve. This is just we want and we may apply all the preceding machinery since we obtain the rational function on the (known) PH-curve but the function itself is a field generator for the new one. If genus of Ξ is nontrivial we obtain new nontrivial ${}_2F_1$ -integrable Fuchsian equations with correct accessory parameters. Let us illustrate the aforesaid. The examples that follow can be enlarged from a large collection of solutions listed in works [11], [12], and [47].

Example 7. Consider a non-PH-solution obtained by P. Boalch in work [11]:

$$y = \frac{7T^2 + 22T + 7}{8(T^2 + T + 1)(T + 2)T}, \quad x = \frac{2T + 1}{(T + 2)T^3}. \quad (91)$$

Correlate it with Picard’s solution (52) (changing there $y \mapsto \tilde{y}$) which is parametrized as follows

$$\tilde{y} = -3 \frac{(\tilde{T} - 3)(\tilde{T} + 1)}{(\tilde{T} + 3)^2}, \quad x = \frac{(\tilde{T} + 1)(\tilde{T} - 3)^3}{(\tilde{T} - 1)(\tilde{T} + 3)^3} \quad \Rightarrow \quad \tilde{T} = \frac{\tilde{y}^3 - 3x\tilde{y} + x(x + 1)}{x(x - 1)}. \quad (92)$$

Equating the x -parts of (91) and (92) to each other, we get $\tilde{T} = \frac{3T-1}{T-1}$. Therefore rational function $T = R(x, y)$ on Boalch’s curve (91) (expression is too large to display here) satisfies the following Fuchsian equation

$$[T, \tau] = \frac{-2(T^2 + T + 1)^4}{(T^2 - 1)^2(2T + 1)^2(T + 2)^2T^2} \quad (93)$$

The simplest way to obtain this equation is to apply Lemma 1 to (9) with the second formula in (91). Equation (93) defines universal uniformizing Hauptmodul for ‘parabolic’ points $T = \{0, \pm 1, -2, -\frac{1}{2}, \infty\}$, but this Hauptmodul is in fact not new because transformation $T \mapsto z$ of the form

$$z = 3 \frac{T + 1}{T - 1}$$

turns (93) into (57) so that we come back to the known consequence of Picard’s curve (52). We obtain, however, Boalch’s y as a rational function on Picard’s curve (92):

$$y = \frac{1}{16} \frac{15\tilde{y}^3 - (14\tilde{y}^2 + 3\tilde{y} - 18)x}{(\tilde{y}^2 - 3\tilde{y} + 3)\tilde{y}}.$$

This gives uniformizing $y(\tau)$ -representation (16) which is not obvious a priori. Indeed, uniformization of algebraic non-PH-solutions is as yet unknown because it does not follow automatically from the PH-series.

In other words, the ‘hyperelliptic’ Hauptmodul $z(\tau)$ defined by (56), (57), or by Proposition 5 is not merely a nice example but exhibits some general recipe.

- Hauptmoduln coming from the PH-series link the zero genus Painlevé curves related to each other through Okamoto’s transformations that, however, *fall outside the scope of pure PH-transformations* (26).

This construction can be continued with involving new curves. Here is a less simple example leading to a non-rational curve.

Example 8. Consider parametrization of the three-branch tetrahedral Hitchin’s solution corrected by Boalch as it has been written in [11, formula (10)]:

$$y = \frac{(T-1)(T+2)}{(T+1)T}, \quad x = \frac{(T-1)^2(T+2)}{(T+1)^2(T-2)}.$$

Parameters for this solution are as follows

$$\alpha = \frac{1}{9}, \quad \beta = 4c^2, \quad \gamma = c^2, \quad \delta = c^2 - \frac{1}{2}, \quad \forall c.$$

Formula for $x(T)$ and Lemma 1 imply

$$[T, \tau] = -\frac{1}{2} \frac{T^6 + 6T^4 - 15T^2 + 44}{(T^2 - 1)^2(T^2 - 4)^2 T^2}.$$

This Hauptmodul is of course universal but corresponds to the set of five parabolic singularities $T = \{\pm 1, \pm 2, \infty\}$ rather than six ones. The curve $\Xi(T, \tilde{T}) = 0$ binding Picard’s \tilde{T} (92) and T turns out to be an elliptic curve and we easily find

$$\Xi : \quad \tilde{T}^4 + 4(T^3 - 3T) \cdot \tilde{T}^3 + 18\tilde{T}^2 - 27 = 0.$$

Corresponding Picard’s and Boalch’s solutions also lie on the nontrivial elliptic curve

$$(4\tilde{y} - 3)(\tilde{y}^4 + 6\tilde{y} - 3)y^6 - 12\tilde{y}(4\tilde{y} - 3)(\tilde{y}^3 + 3\tilde{y} - 2)y^5 + \dots - 12\tilde{y}^5(\tilde{y}^3 + 3\tilde{y} - 2)y + 4\tilde{y}^8 = 0,$$

where we reduced the formula and designated by dots the polynomials in descending powers of y (exercise: find Boalch’s y and Picard’s \tilde{y} as rational functions on Picard’s and Boalch’s curves respectively). These two curves are of course isomorphic and easy computation shows that they are birationally equivalent to the simple Weierstrass canonical form¹² $v^2 = 4u^3 - 48u + 80$.

The symmetry $T \mapsto -T$ prompts us to consider the Hauptmodul $T = T^2$ and we readily obtain its Fuchsian equation:

$$[T, \tau] = -\frac{3}{8} \frac{1}{T^2} - \frac{1}{2} \frac{1}{(T-1)^2} - \frac{1}{2} \frac{1}{(T-4)^2} + \frac{1}{8} \frac{7T-11}{(T-1)(T-4)T}$$

which defines a thrice punctured sphere at $T = \{1, 4, \infty\}$ with an additional conical singularity of the second order at $T = 0$. Such equations can also be applied to uniformization

¹²Although this torus and torus (53) arise from one PH-curve, they have different J -invariants: $-\frac{16}{9}$ and $\frac{2197}{972}$ respectively. The latter invariant is the same as for curve (76).

(mixed uniformization in terminology of [14]). Solution to inversion problem for T and \mathbf{T} is somewhat nonstandard and we shall present it elsewhere.

Example 9 (exercise). Consider any PH-curves for $N = \{3, 4\}$ and construct the universal Hauptmodul, its differentials, nice Fuchsian ODE

$$[T, \tau] = -\frac{1}{2} \sum_{\alpha} \frac{1}{(T - \alpha)^2} + \frac{4T^6 - 14T^4 + 2}{T^9 - 6T^7 + 6T^3 - T},$$

where $\alpha^9 - 6\alpha^7 + 6\alpha^3 - \alpha = 0$, its ${}_2F_1$ -solutions, and do derive and uniformize the curve $\Xi(T, \tilde{T}) = 0$ of genus $g = 7$ and some elements of its tower. Notice that genera of these PH-curves, according to the table (47), are still equal to zero.

10. HAUPTMODULN AND ANALYTIC CONNECTIONS

Formulae (84)–(88) of Sect. 8.3 were substantially intended just for illustration to an important general construction outlined in the chain (89). This is because the number of independent differential consequences is not infinite; they unify into a whole the base objects of the theory: Hauptmoduln x, z, s, \dots , solutions Ψ (possibly in terms of ${}_2F_1$), and the primary differential $\Psi^2 = \dot{x}$ as a weight-2 modular form. Closer motivation and explanations for that interrelations are as follows.

10.1. Differential structures on Riemann surfaces. Every Riemann surface \mathcal{R} of finite analytic type (both genus and number of punctures are finite [8, 44]) is described by the 2nd order equation (8). Therefore we have the two objects $\psi_1(x)$, $\psi_2(x)$ and their differential closure requires, at most, the two more objects $\psi'_1(x)$, $\psi'_2(x)$. A Wronskian relation between these objects, that is $\psi_1\psi'_2 - \psi_2\psi'_1 = \exp(-\int p dx)$ (the known function), sifts three of them as differentially independent. We may also think of the uniformizing function $x = \chi(\tau)$ as a local coordinate transition¹³ $x \rightleftarrows \tau$. Hence there should exist the single-valued ‘ τ -representations’ for equivalents of these three base differential¹⁴ objects on \mathcal{R} and the \mathcal{R} itself is considered now as a 1-complex-dimensional analytic manifold. Inso-much as all analytic tensor fields T on Riemann surfaces reduce to analytic k -differentials $T(\tau)d\tau^k$, we can construct them from an object defining completely our \mathcal{R} , i.e. the factor topology on $\overline{\mathbb{H}^+/\mathfrak{G}}$. This is of course the fundamental function $\mathcal{Q}(x, y)$ because it defines the above mentioned ψ and ψ' in ‘ x -representation’. It also generates the scalar (automorphic function) $x(\tau)$ on this \mathcal{R} through the \mathfrak{D} -derivative $[x, \tau] = \mathcal{Q}(x, y)$; any other equivalent Hauptmodul has the form $\frac{Ax+B}{Cx+D}$. The simplest 1-tensor (call it g_1) is an analytic covariant and general way of building such an object is to take a meromorphic differential $g_1 = R(x, y)dx$ (our interest now is a meromorphic analysis on \mathcal{R}). The latter, as an Abelian differential, is always the primary differential $\dot{x}(\tau)$ (weight-2 automorphic form) multiplied by any other scalar: $g_1(\tau) = R(x, y)\dot{x}$. In the sections that follows we shall restrict our consideration to the case $\mathcal{Q} = \mathcal{Q}(x)$ (genus $g = 0$) and higher genera will be considered elsewhere.

¹³Of course, with a corresponding change of complex structure preserving the \mathcal{R} itself.

¹⁴We speak here of only analytic (not metric!) objects since \mathcal{R} is already assumed to be given.

Motivated by a desire to build the differential apparatus on \mathcal{R} , we need to use derivatives of the Hauptmodul-scalar on \mathcal{R} in order to introduce the covariant differentiation $\nabla = \partial_u - \Gamma(u)$ with the standard transformation law $\Gamma \mapsto \tilde{\Gamma}$:

$$\tilde{\Gamma}(\tilde{u})d\tilde{u} = \Gamma(u)du - d\ln \frac{d\tilde{u}}{du}.$$

There are many realizations of complex structures $u \mapsto \tilde{u}$ for a given \mathcal{R} but universal cover for all finite \mathcal{R}' is \mathbb{H}^+ ; this being so, we put $u \in \mathbb{H}^+$ and, renaming $u \mapsto \tau$, impose the projective structure $\tau \mapsto \tilde{\tau} = \frac{a\tau+b}{c\tau+d}$, where $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{PSL}_2(\mathbb{R})$. It follows that we have to construct the connection object $\Gamma(\tau)$ that, as a function on \mathbb{H}^+ , respects the factor topology on $\overline{\mathbb{H}^+/\mathfrak{G}_x}$ and behaves as an affine connection does:

$$\tilde{\Gamma}(\tilde{\tau}) = (\mathbf{c}\tau + \mathbf{d})^2 \Gamma(\tau) + 2\mathbf{c}(\mathbf{c}\tau + \mathbf{d}). \quad (94)$$

On the other hand, we know that any scalar object $x = \chi(\tau)$ on arbitrary \mathcal{R} is defined by the two structure properties

$$x(\tilde{\tau}) = x\left(\frac{a\tau+b}{c\tau+d}\right) \quad \text{and} \quad \chi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \chi(\tau) \quad \forall \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{G}_x \quad (95)$$

(we supposed that $\pi_1(\mathcal{R}) = \mathfrak{G}_x$). Hence it follows that the Γ -object can be built with the help of differential or scalar by the rule

$$\Gamma(\tau) = \frac{d}{d\tau} \ln g_1 = \frac{d}{d\tau} \ln R(x) \dot{x}(\tau).$$

To put it differently, we are interested in Γ 's compatible with scalars on \mathcal{R} .

The above-mentioned differential closure is already available because the object $\{x, \tau\}d\tau^2$ becomes (as it follows from (10)) a tensor when complex structure is a projective one. Hence $\{x, \tau\}d\tau^2 = \mathcal{Q}(x)dx^2$ and, finally, the transition between x - and τ -representations can be represented in the form of the following equivalence

$$\{\psi_1(x), \psi_2(x), \psi'_1(x)\} \Leftrightarrow \{\chi(\tau), \dot{\chi}(\tau), \Gamma(\tau)\},$$

where

$$\tau = \frac{\psi_2(x)}{\psi_1(x)}, \quad \dot{\chi}(\tau) = \mathbf{E}\psi_1^2(x), \quad \Gamma(\tau) = 2\mathbf{E}\psi_1\psi'_1 + (p + \ln' R)\mathbf{E}\psi_1^2 \quad (96)$$

and $\mathbf{E} := \exp \int pdx$; if the normal form (5) is chosen then one puts here $(p, \mathbf{E}) = (0, 1)$. By this means the complete set of data for the analytic theory on \mathcal{R} is defined by the relations (96) and is described by the *two* (not one) *fundamental ODEs* for the ψ -function and its derivative $\psi' =: \Phi$:

$$\begin{aligned} \psi'' + p\psi' + q\psi &= 0, \\ \Phi'' + (p - \ln' q)\Phi' + (q + p' - p\ln' q)\Phi &= 0. \end{aligned} \quad (97)$$

The functions $R(x)$, under this setting, calibrate all the Γ 's. Of course, we may (for convenience sake) freely change the basis $\{\psi_1, \psi_2, \Phi_1, \Phi_2\}$ over \mathbb{C} or $\mathbb{C}(x)$ and rewrite (in an equivalent way) the theory in terms of the four 1st order ODEs. The formulae (85)–(88), for example, illustrate this construction when group is $\Gamma(2)$.

Remark 8. The geometric treatment to the famous Chazy equation $\pi \ddot{\eta} = 12i(2\eta \ddot{\eta} - 3\dot{\eta}^2)$ as one defining the Γ -connection for group $\Gamma(1)$ was given by Dubrovin in lectures [23]. In the same place he showed that other equations found by Chazy fit also in such a scheme. See Appendix C of [23] for more examples and voluminous references. The following simple example shows that Dubrovin's $\Gamma(1)$ -connection is compatible with scalar.

Example 10. In the case of group $\Gamma(1)$ we may take $x = J(\tau)$. The simplest idea for a basic weight-2 automorphic form g_1 is to take the ratio of modular Weierstrass' forms $g_1 = \frac{g_3}{g_2}$. In terms of Hauptmodul we therefore should put

$$J(\tau), \quad g_1(\tau) := \frac{\pi i}{36} \frac{J}{J}, \quad \Gamma(\tau) := \frac{\ddot{J}}{J}. \quad (98)$$

As a references source we give relations to the standard functions, i.e. representations for standard *holomorphic* forms on $\Gamma(1)$ through the objects (98):

$$g_2(\tau) = \frac{27J}{J-1} g_1^2, \quad g_3(\tau) = \frac{27J}{J-1} g_1^3, \quad \eta(\tau) = \frac{\pi}{4i} \Gamma + \frac{3}{2} \frac{7J-4}{J-1} g_1$$

and

$$\nabla g_1 = (\partial_\tau - \Gamma) g_1 = \frac{6}{\pi i} \frac{J+2}{J-1} g_1^2.$$

The connection Γ is not holomorphic everywhere in \mathbb{H}^+ but all the Γ 's are defined up to a 1-differential. Subtracting/adding a certain differential we obtain the holomorphic connection

$$\Gamma \rightarrow \Gamma - \frac{1}{6} \frac{d}{d\tau} \ln J^4 (J-1)^3 = \frac{4i}{\pi} \eta(\tau). \quad (99)$$

Remark 9 (exercise). Do write down the $\Gamma(1)$ -equations (97) and complete set of their solutions in terms of special functions (not merely in terms of ${}_2F_1$ -series). Derive direct/inverse modular transitions between $\{\psi(x), \Phi(x)\}$ and $\{\tau\}$. Curiously, the group $\Gamma(1)$ is not less known than $\Gamma(2)$ but no formulae solving this exercise (analog of (85)–(88)) have appeared in the literature.

10.2. ODEs satisfied by analytic connections on \mathcal{R} 's. Since the general classes of Dubrovin's Γ -equations [23, p. 152, **Exercise C.3**] respect the most general projective structure $GL_2(\mathbb{C})$, their discrete symmetries (if any) must not necessarily be Fuchsian. Without any specification these may formally be non-finitely generated groups or described even by not Fuchsian equations¹⁵ (to say nothing of monodromies of Fuchsian type). In other words, projectively correct Γ -equations do not touch on the question of finiteness of genus and compatibility of connections with Hauptmoduln-scalars, whereas any finite \mathcal{R} 's are associated, as described above, with such Γ 's (hence Dubrovin's Γ 's as well) and we have actually had now a large number of examples with their monodromies known to be Fuchsian.

¹⁵An example: formula (C.44) in **Exercise C.4** on p.155 in [23].

Theorem 10. *Let \mathfrak{T} be a genus zero uniformizing orbifold defined by Fuchsian equation $[x, \tau] = \mathcal{Q}(x)$ having only parabolic singularities $x = E_k$. Then the expression*

$$\Gamma = \frac{d}{d\tau} \ln \dot{x} \quad (100)$$

is an analytic connection (respecting the factor topology on $\overline{\mathbb{H}^+/\mathfrak{G}_x}$) everywhere holomorphic on $\mathbb{H}^+ \cup \{\infty\}$. It satisfies a 3rd order polynomial ODE $\Xi(\dot{\Gamma}, \ddot{\Gamma}, \ddot{\Gamma}, \Gamma) = 0$ with constant coefficients. Any other analytic connection also satisfies an ODE of such a kind. General solutions to these equations are constructed by the scheme

$$\Gamma(\tau) \mapsto \frac{\Gamma\left(\frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^2} - \frac{2c}{c\tau+d}. \quad (101)$$

Proof. The statement about factor topology is obvious from the property (95) of $x = \chi(\tau)$ to be a scalar. Since \mathfrak{T} is an orbifold of finite genus, the existence domain of $\chi(\tau)$ is an interior of a circle or a half-plain. Normalize this domain to be \mathbb{H}^+ . For all $\tau, \tau_0 \in \mathbb{H}^+$ we have a convergent series representation $\chi(\tau) = x_0 + a(\tau - \tau_0) + \dots$, where $a \neq 0$. It follows that (100) is holomorphic everywhere in \mathbb{H}^+ . For infinite point we make the standard change $\tau \mapsto q$ of the local parameter $q = e^{\pi i \tau}$, where $\tau \rightarrow i\infty$. Then we may write $x = E + aq^n + \dots$, where E , a , and $n \in \mathbb{Z}$ depend on the local monodromy \mathfrak{G}_x . Taking into account that $dq = \pi i q d\tau$ we derive that (100) is again holomorphic as $q \rightarrow 0$. Zeroes/poles and behavior of connection on the real axis are not well defined since its transformation law is inhomogeneous.

Let us denote $\Omega := \dot{\Gamma} - \frac{1}{2}\Gamma^2$. The equation $[x, \tau] = \mathcal{Q}(x)$ determining the scalar x becomes the tensor $\mathcal{Q} \cdot \dot{x}^2 = \Omega$. Applying ∇ , we get the two equations:

$$\Omega = \mathcal{Q} \cdot (\nabla x)^2, \quad \nabla \Omega = \nabla \mathcal{Q} \cdot (\nabla x)^2 = \mathcal{Q}' \cdot (\nabla x)^3, \quad (102)$$

since $\nabla x = \dot{x}$ and $\nabla \dot{x} \equiv 0$. Elimination of ∇x gives the identity

$$\frac{(\nabla \Omega)^2}{\Omega^3} = \frac{\mathcal{Q}'}{\mathcal{Q}^3}. \quad (103)$$

The second covariant derivative $\nabla^2 \Omega = \mathcal{Q}'' \cdot (\nabla x)^4$ yields yet another identity:

$$\frac{\nabla^2 \Omega}{\Omega^2} = \frac{\mathcal{Q}''}{\mathcal{Q}^2} \quad (104)$$

and ∇ -derivatives are understood here to be equal to

$$\nabla \Omega := \dot{\Omega} - 2\Gamma \Omega, \quad \nabla^2 \Omega := (\partial_\tau - 3\Gamma)(\partial_\tau - 2\Gamma)\Omega. \quad (105)$$

Since identities (102), (103) are of invariant (scalar) type, the sought-for Γ -equation results from elimination of x . This will be a 3rd order ODE $\Xi(\Omega, \nabla \Omega, \nabla^2 \Omega) = 0$ with constant coefficients. Notice incidentally that this equation still holds if group \mathfrak{G}_x is not a 1st kind Fuchsian one, i.e. when $\mathcal{Q}(x)$ has non-correct ('bad') accessory parameters.

Let γ be any other connection. Then $\Gamma = \gamma + R(x)\dot{x}$, where $R(x)\dot{x}$ is a certain 1-differential. For example, if original Fuchsian equation has non-canonical form (8),

it is convenient to put $R(x) = p(x)$, that is $\Gamma = \gamma + p(x)\dot{x}$. Hence we redefine Ω as

$$\Omega = \partial_\tau(\gamma + R\dot{x}) - \frac{1}{2}(\gamma + R\dot{x})^2 = \left(\dot{\gamma} - \frac{1}{2}\gamma^2\right) + \left(R' + \frac{1}{2}R^2\right)\dot{x}^2,$$

since $\ddot{x} = (\gamma + R\dot{x})\dot{x}$. Denoting $\omega := \dot{\gamma} - \frac{1}{2}\gamma^2$ one gets

$$\omega = \left(\mathcal{Q} - R' - \frac{1}{2}R^2\right)\cdot\dot{x}^2 \quad \Rightarrow \quad \omega = Q \cdot (\tilde{\nabla}x)^2. \quad (106)$$

where $Q := \mathcal{Q} - R' - \frac{1}{2}R^2$ and $\tilde{\nabla}$ signifies the differentiation by means of connection γ . Clearly, x is a ‘flat’ coordinate only with respect to the ‘old’ connection (100), i.e. $\tilde{\nabla}^2x \not\equiv 0$ now, and we have $\tilde{\nabla}^2x = R \cdot (\tilde{\nabla}x)^2$ instead of $\nabla^2x \equiv 0$. Therefore $\tilde{\nabla}$ -derivatives of (106) give the two identities

$$\tilde{\nabla}\omega = (Q' + 2RQ)(\tilde{\nabla}x)^3, \quad \tilde{\nabla}^2\omega = (Q'' + 5RQ' + 2R'Q + 6R^2Q)(\tilde{\nabla}x)^4$$

which are the generalizations of Eqs. (103)–(104):

$$\frac{(\tilde{\nabla}\omega)^2}{\omega^3} = \frac{(Q' + 2RQ)^2}{Q^3}, \quad \frac{\tilde{\nabla}^2\omega}{\omega^2} = \frac{1}{Q^2}(Q'' + 5RQ' + 2R'Q + 6R^2Q).$$

As before, the equation $\Xi(\ddot{\gamma}, \ddot{\gamma}, \dot{\gamma}, \gamma) = 0$ follows by elimination of x and

$$\tilde{\nabla}\omega := (\partial_\tau - 2\gamma)\left(\dot{\gamma} - \frac{1}{2}\gamma^2\right), \quad \tilde{\nabla}^2\omega := (\partial_\tau - 3\gamma)(\partial_\tau - 2\gamma)\left(\dot{\gamma} - \frac{1}{2}\gamma^2\right). \quad (107)$$

Transformation law (94) for all the Γ ’s entails the formula (101) for solutions to these equations wherein $\Gamma(\tau)$ is any particular solution, say (100), if Hauptmodul $x = \chi(\tau)$ has been given. \blacksquare

Notice that in the canonical case $(p, R, \mathbf{E}) = (0, 1, 1)$ the logarithmic derivative $2\ln_\tau\Psi$ satisfies the same equation as Γ does; according to the 3rd formula in (96), we have

$$\Gamma(\tau) = 2\Psi\Psi' = 2\frac{\Psi_\tau}{\Psi}$$

(this is a simplest motivation for introducing the derivative of the Ψ -function). It should be noted here that the 3rd order γ -ODE may turn out to be simpler if the γ -definition corresponds not to \mathfrak{G}_x but to a wider group. For example, the connection

$$\gamma = \frac{d}{d\tau}\ln\dot{s} - \frac{3}{2}\frac{s-5}{s(s-9)}\dot{s},$$

constructed formally as one on the Heun group \mathfrak{G}_s (83), is in fact a hidden form of the $\Gamma(1)$ -connection satisfying the equation $\ddot{\gamma} = 6\gamma\ddot{\gamma} - 9\dot{\gamma}^2$ (proof is a calculation with use of (74) and (99)). In other words, *connections, along with uniformizing Hauptmoduln $x = \chi(\tau)$, also form towers according to a tower of subgroups*.

Example 11. The connection for Legendre’s modulus $k(\tau) = \frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)}$ with monodromy \mathfrak{G}_u defined by equation (49). From Theorem 10 we obtain the following Γ -equation:

$$\begin{aligned} A^8 - 8\Omega(B - 352\Omega^2)A^6 + 24\Omega^2(B^2 - 260\Omega^2B - 368\Omega^4)A^4 \\ - 32\Omega^3(B^3 - 129\Omega^2B^2 - 168\Omega^4B + 944\Omega^6)A^2 + 16\Omega^4(B^2 - 20\Omega^2B - 80\Omega^4)^2 = 0, \end{aligned}$$

where $A := \nabla\Omega$, $B := \nabla^2\Omega$, and Ω as in (105) and

$$\Gamma = \frac{d}{d\tau} \ln \frac{d}{d\tau} k(\tau) = \frac{4i}{\pi} \eta + \frac{1}{6} \pi i (\vartheta_3^4 - 5\vartheta_2^4).$$

Renormalization of this Γ into connection

$$\begin{aligned} \Gamma \rightarrow \gamma &= \Gamma - \frac{d}{d\tau} \ln k(k^2 - 1) = \frac{d}{d\tau} \ln \vartheta_2^4 \\ &= \frac{4i}{\pi} \eta + \frac{1}{3} \pi i (2\vartheta_2^4 - \vartheta_3^4), \end{aligned}$$

which corresponds to original Chudnovsky equation (I), yields more compact equation:

$$(2\dot{\gamma} - \gamma^2) \ddot{\gamma} = 2\ddot{\gamma}(\ddot{\gamma} - \gamma^3) - \dot{\gamma}^2(2\dot{\gamma} - 3\gamma^2).$$

Example 12. The γ -equation for Chudnovsky equation (II). It admits the compact form

$$2\omega (\tilde{\nabla}^2\omega + 6\omega^2)^2 = (2\tilde{\nabla}^2\omega + 15\omega^2)(\tilde{\nabla}\omega)^2,$$

where $\tilde{\nabla}\omega$, $\tilde{\nabla}^2\omega$ are determined by (107) and the scalar compatible γ is defined here as

$$\gamma = \frac{d}{d\tau} \ln \dot{x} - \frac{d}{d\tau} \ln \{(x+1)^3 - 1\}.$$

We do not display here formulae related to most interesting equation (66) since its Γ , γ -equations have somewhat cumbrous form (exercise: list these ODEs for all Chudnovsky's equations). It is easily derivable but we note in passing that these examples, i.e. \mathfrak{G}_u and \mathfrak{G}_s , show that the direct search for Γ -equations in form of the tensor $\{\Omega, \nabla\Omega, \nabla^2\Omega, \dots\}$ -Ansätze [23] can be very difficult problem. Moreover, normalization of connections by differential $R(x)$ significantly affects the size of equation. For example, the natural but 'unlucky' Γ -definition (98) leads to useless equations like

$$2642368542992676B^6 - 264634415204382300B^5\Omega^2 + \dots + 26676039242547325440000\Omega^{12}$$

with the same meaning for A , B , and Ω as above.

If Hauptmodul has elliptic singularities then construction of the holomorphic connection is performed by a simple reproducing of what we have done in Example 10, that is (99). Let zero genus Hauptmodul $x(\tau)$ have an N -order conical point $x = C$; that is $x = C + a(\tau - \tau_0)^N + \dots$. In order to compensate singularities coming from fold-points of $x(\tau)$ we add to connection (100) the differential $\frac{1-N}{N} \frac{\dot{x}}{x-C}$ for each such a point. The Γ remains regular at infinite point.

Proposition 6. *Everywhere in $\mathbb{H}^+ \cup \{\infty\}$ holomorphic affine connection for a zero genus orbifold defined by the Fuchsian equation $[x, \tau] = \mathcal{Q}(x)$ with N_k -order elliptic singularities at points $x = C_k$ is determined by the expression:*

$$\Gamma = \frac{d}{d\tau} \ln \dot{x} - \sum_k \frac{N_k - 1}{N_k} \frac{d}{d\tau} \ln(x - C_k).$$

Although connections are not uniquely defined we can remove the ambiguity by normalizing location of the three points $x = \{0, 1, \infty\}$ of a Hauptmodul at fixed cusps.

Study of nonlinear differential equations associated with certain modular forms and generalizations of Chazy's equations were initiated by M. Ablowitz et all in the nineties in connection with mathematical physics problems including magnetic monopoles [38], self-dual Yang–Mills and Einstein equations [36], as well as topological field theories [23]. Recently they again attracted attention [1] and the nice work [49] by R. Maier provides explicit examples related to some low level groups $\Gamma_0(N)$, hypergeometric equations, and number-theoretic treatments. The works [1, 49] provide also voluminous references along these lines.

We left some remarks and examples in this work as exercises because the stream of consequences, including hyperelliptic, may be increased considerably. The abundance of towers, Hauptmoduln, connections, ODEs, and groups requires their further classification, and with it the unification of ways of getting the θ -formulae and groups.

11. REMARKS ON ABELIAN INTEGRALS AND EQUATIONS ON TORI

Described ‘differential θ -machinery’ makes it possible to include immediately into analysis Abelian integrals: holomorphic, meromorphic, and logarithmic. Insomuch as many of the curves we have considered (probably all) admit representation in form of covers over tori, we can derive a large family of explicitly solvable Fuchsian equations on tori; in doing so fundamental logarithmic Abelian integral and all the meromorphic objects on our Riemann surfaces can be described in terms of Jacobi's θ -functions and constants. As we have already mentioned in Introduction, no such representations are presently available in spite of numerous examples of modular curves and the well-studied subgroups of $\Gamma(1)$. We touched briefly on this problem in [14] and exhibited there one special example of a meromorphic integral. Abelian integrals, Fuchsian equations on tori, and higher genus integer q -series certainly merit thorough investigation and will be fully considered in a continuation of this work. Anticipating such a kind results by explicit formulae, we consider briefly the hyperelliptic series (59), in particular, equation (90).

It is common knowledge that all such curves admit a reduction of the holomorphic hyperelliptic integrals into elliptic ones. In particular the reduction formula for curve (90) has the form

$$\wp(\mathbf{u}) = \frac{(z \pm \varkappa)^2}{(z \mp \varkappa)^2} \quad (\varkappa := i\sqrt{3}) \quad (108)$$

which entails the relation $\wp'^2(\mathbf{u}) = 4\wp^3(\mathbf{u}) - 4$; recall that $z(\tau)$ satisfies (57). It follows (Lemma 1) that the two nice formulae describe the theory of this curve:

$$\sqrt{\mp\varkappa} \cdot d\mathbf{u} = \frac{(z \pm \varkappa)dz}{\sqrt{z^5 - 10z^3 + 9z}}, \quad [\mathbf{u}, \tau] = -6 \frac{2\wp^3(\mathbf{u}) + 1}{\wp^2(\mathbf{u})\wp'^2(\mathbf{u})}$$

(these and subsequent computation details will be presented elsewhere). Specify what does this mean. Both the holomorphic integrals $\mathbf{u}^\pm = \mathbf{u}(\tau)$ defined by the equations above are the globally single-valued analytic functions on \mathbb{H}^+ because equation (108) is obviously none other than an equivalent of the Riemann surface defined by hyperelliptic form (90).

As for equations on tori we exhibit a nice example of a singly punctured torus

$$[\mathfrak{u}, \tau] = -2\wp(2\mathfrak{u}) - \frac{8}{3} \quad \Leftrightarrow \quad [\mathfrak{u}, \tau] = -2\wp(2\mathfrak{u}|\varepsilon) - \frac{4760}{1971} \frac{g_3(\varepsilon)}{g_2(\varepsilon)}, \quad (109)$$

where $\mathfrak{u} = \omega\mathfrak{u}$ and Weierstrass' $\wp(\mathfrak{u}) = \wp(\mathfrak{u}; a, b) = \wp(\mathfrak{u}|\omega, \varepsilon\omega)$ corresponds to invariants $(a, b) = \left(\frac{292}{3}, \frac{4760}{27}\right)$ and therefore $J(\varepsilon) = \frac{73^3}{243^2}$ [17, p. 185]¹⁶. Monodromy of this equation is generated by the two transformations $\mathfrak{a}(\mathfrak{u}) = \mathfrak{u} + 1$ and $\mathfrak{b}(\mathfrak{u}) = \mathfrak{u} + \varepsilon$; the loop around $\mathfrak{u} = 0$ is equivalent to the transformation $\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$. The result of the theory is that this equation and (66) are turnable one to another by the very simple substitution

$$s = \wp(\mathfrak{u}) + \frac{10}{3} \quad (110)$$

and therefore the global monodromy \mathfrak{G}_u is an index two 2-generated subgroup of Heun–Apéry's one \mathfrak{G}_s described in Theorem 9. By construction, the solution $\mathfrak{u}(\tau)$ to equation (109) is an additively automorphic and globally single-valued analytic function on \mathbb{H}^+ . An explicit representation to $\mathfrak{u}(\tau)$ is in fact the latter formula

$$\mathfrak{u}(\tau) = \wp^{-1} \left(9 \frac{\vartheta_3^4(3\tau)}{\vartheta_3^4(\tau)} - \frac{10}{3} \right).$$

Remembering now that $s = z^2$, we may write an equivalent of substitution (110):

$$z^2 = \wp(\mathfrak{u}) + \frac{10}{3} \quad (111)$$

which in turn coincides with a simplest nontrivial ($g > 0$) example of covers over tori, i.e. representations of special but very wide class of \mathcal{R} 's. These have the form $z^2 = \wp(\mathfrak{u}) - e$, where e is any of standard Weierstrass' branch points. Since $\wp(\varepsilon\omega) = -\frac{10}{3}$, the cover (111) is of unit genus. It is isomorphic to the torus (53) having invariant $J = \frac{2197}{972}$.

Yet another remarkable consequence of arisen ‘toroidal covers’ is a corollary of the equivalence of two Chudnovsky's equations under the substitution (76). It follows at once that representation of these two orbifolds through punctured tori will yield a nontrivial representation for the mutually transcendental covers $\Xi(\mathfrak{p}, \mathfrak{u}) = 0$ of the punctured torus (109) and a torus defined by Chudnovsky's equation (75). The latter has long been known [41, 17] and has the form of a ‘punctured lemniscate’ $[\mathfrak{p}, \tau] = -2\wp(2\mathfrak{p}|\mathfrak{i})$. It is transformed into Chudnovsky's equation (75) by the further simple substitution $\mathfrak{x} = \wp(\mathfrak{p}|\mathfrak{i})$ (exercise: check this). Correlating this substitution, (110), and (67) with equation (76), we get the sought-for function $\Xi(\mathfrak{p}, \mathfrak{u})$:

$$\frac{(\wp(\mathfrak{u}; a, b) + \frac{1}{3})^2 - 12}{2\wp^2(\mathfrak{p}; 4, 0) - 1} = \sqrt{8} \frac{\pi}{\omega} \vartheta_2\left(\frac{\varepsilon}{2}\right) \cdot \frac{\theta_4\left(\frac{\mathfrak{u}}{2\omega}|\varepsilon\right)}{\theta_1\left(\frac{\mathfrak{u}}{2\omega}|\varepsilon\right)}, \quad \omega^2 := \frac{657}{1190} \frac{g_3(\varepsilon)}{g_2(\varepsilon)} \quad (112)$$

¹⁶We failed to find out in reference books on imaginary quadratic number fields (e.g. [62]) an exact value for $\varepsilon \approx 1.5634019226921973634612986241 \cdot i$. This torus is perhaps exceptional indeed.

($\omega \approx 0.5391289118749108088596687497$). To put it differently, as can not well be imagined, the two ‘simple’ Fuchsian equations

$$[\mathfrak{u}, \tau] = -2\wp(2\mathfrak{u}; a, b) - \frac{8}{3}, \quad [\mathfrak{p}, \tau] = -2\wp(2\mathfrak{p}; 4, 0)$$

are transformable into each other by the transcendental and highly non-obvious cover (112). In this regard they are also integrable along with the four Chudnovsky’s equations (Corollary 3). The cover (112) is of course completely uniformized by single-valued functions $\mathfrak{u}(\tau)$ and $\mathfrak{p}(\tau)$. It is a very nontrivial exercise to investigate it directly, i.e. to describe its branch schemes, Puiseux series, and compute the genus that is unity.

What we have done now is in effect the draft recipes of getting the large number of explicit formulae including the very nontrivial ‘toroidal towers’. But all this is a corollary of just simplest PH-curves.

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